Genus one String Amplitudes from Conformal Field Theory

Luis Fernando Alday

University of Oxford

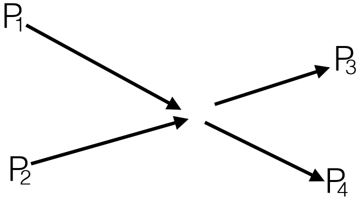
String Math 2019 - Uppsala

What will this talk be about?

Conformal Field Theories techniques to study Scattering Amplitudes in theories of Gravity/String Theory.

Based on work with A. Bissi and E. Perlmutter.

Scattering Amplitudes: Probability that two particles colliding (with momenta p_1, p_2) result into two other particles (with momenta p_3, p_4).



$$= \quad A(g, s, t, u)$$

- $A(g, s, t, u)$ depends on many things:
  - Which particles you are scattering (their masses, charges, etc)
  - The parameters of your theory g.
  - The momenta of the particles being scattered:

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2, \quad u = -(p_1 - p_4)^2$$

Why scattering amplitudes?

- They can teach us much about the symmetries a theory.
  - Dual conformal symmetry.
  - Duality with Wilson loops.
  - Color/kinematic duality.
- Beautiful mathematical structures can be uncovered.
  - Twistor methods.
  - The positive grassmanian.
- They allow to test the predictions of our theory.

Scattering amplitudes in pure gravity

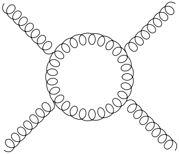
General Relativity: Einstein Hilbert Lagrangian

$$\mathcal{L}_{EH}[g] = \frac{1}{G_N}\sqrt{-g}\mathcal{R}$$

Consider a scattering process: $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G_N} h_{\mu\nu}$

$$\mathcal{L}_{EH} = (\partial h)^2 + \sqrt{G_N} h(\partial h)^2 + ...$$

Scattering of gravitons:

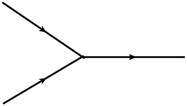


- UV divergences: the gravitons get too close to each other!

String theory and UV divergences

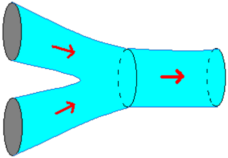
- Replace gravitons by closed strings of finite length $\sqrt{\alpha'}$

Point particles



VS

Strings



- At low energies, $p^2 \ll 1/\alpha'$ we recover GR (or SUGRA).

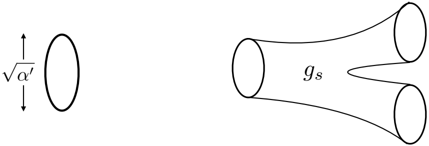
GR as an effective field theory

$$\mathcal{L} = (\partial h)^2 + \sqrt{G_N}\left(h(\partial h)^2 + \alpha'^{1/2} h^2 (\partial h)^2 + \alpha' h (\partial h)^3 + \cdots\right)$$

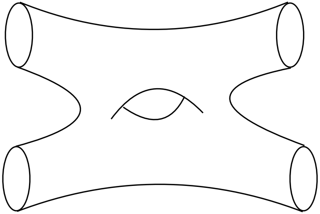
- At high energies stringy corrections give a UV completion of GR!

String theory scattering amplitudes

- In string theory we have two parameters

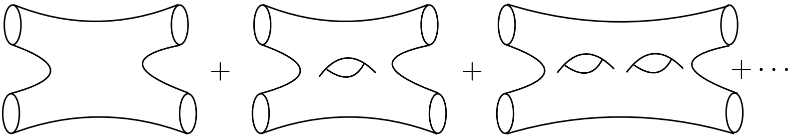


- We would like to compute scattering amplitudes



$$= A(g_s, \alpha', s, t, u)$$

- The computation organises in a genus expansion

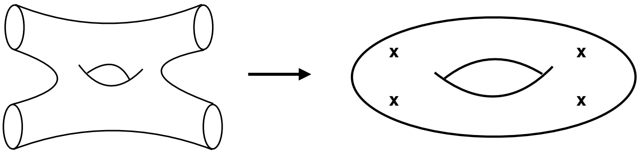


$$A^{(genus\ 0)}(\alpha', s, t, u) + g_s^2 A^{(genus\ 1)}(\alpha', s, t, u) + g_s^4 A^{(genus\ 2)}(\alpha', s, t, u) + \cdots$$

At genus zero and in flat space: Virasoro-Shapiro amplitude

$$A^{(genus\ 0)}(\alpha', s, t, u) = \frac{\Gamma(-\frac{\alpha' s}{4})\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})}$$

- Even in flat space higher genus terms are notoriously hard to compute!



$$A^{(genus\ 1)}(\alpha', s, t, u) = \int_{\mathcal{F}} \frac{d\tau_1 d\tau_2}{\tau_2^2} F(s,t,u;\tau)$$

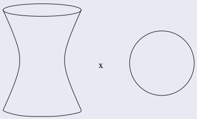
$$F(s,t,u;\tau) = \int d\nu_1 d\nu_2 d\nu_3 d\nu_4 \left| \frac{\theta_1(\nu_1-\nu_2|\tau)}{\theta_1'(0|\tau)} \right|^{2\alpha' s} \cdots \left| \frac{\theta_1(\nu_2-\nu_4|\tau)}{\theta_1'(0|\tau)} \right|^{2\alpha' u}$$

- For curved space time, we have not idea how to do this!
- We will use an alternative approach! (based on two tools)

First tool

AdS/CFT duality

String theory on $AdS_5 \times S^5$ $\Leftrightarrow$ Quantum field theory living in the 4d boundary of AdS_5

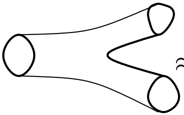


- $\mathcal{N}=4$ Super conformal Yang-Mills, with $SU(N)$ gauge group.
- Most symmetric 4d theory!

Parameters: g_s and R.

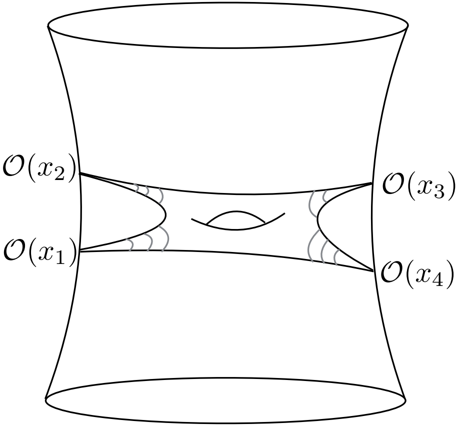
Pameters: g_{YM} and N.

Dictionary

$g_s =$  $\approx \frac{1}{N}$, $\frac{R^2}{\alpha'} = \sqrt{g_{YM}^2 N} \equiv \sqrt{\lambda}$

AdS/CFT duality

String amplitudes on $AdS_5 \times S^5 \leftrightarrow$ correlators of local operators!



$$A(g_s, \frac{\alpha'}{R^2}, s, t, u) \leftrightarrow \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle$$

Dictionary

String Amplitudes on $AdS_5 \times S^5$	Correlators in $\mathcal{N}=4$ SYM
Genus expansion	$1/N$ expansion
Stringy corrections to Sugra	$1/\lambda$ corrections
Graviton on AdS	$\mathcal{O}_2$: protected scalar of dim. 2 in the stress-tensor multiplet
KK modes on S^5	$\mathcal{O}_p$: protected ops of dim. p

- Compute $\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$ in a $1/N$ and $1/\lambda$ expansion.
- Thanks to our second tool: the analytic bootstrap!

CFT basics

Main ingredient:

- Conformal Primary local operators: $\mathcal{O}_{\Delta,\ell}(x)$, plus descendants $\partial_{\mu_k}...\partial_{\mu_1}\mathcal{O}_{\Delta,\ell}$

Dimension — Lorentz spin

Main observable:

Correlation functions of primary operators

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle$$

Four-point function of identical operators:

$$\boxed{\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{\mathcal{G}(u,v)}{x_{12}^{2\Delta_\mathcal{O}}x_{34}^{2\Delta_\mathcal{O}}}}$$

where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

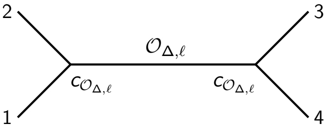
CFT basics

Crossing symmetry

$$v^{\Delta_{\mathcal{O}}}\mathcal{G}(u,v) \underbrace{=}_{x_1\leftrightarrow x_3} u^{\Delta_{\mathcal{O}}}\mathcal{G}(v,u) \underbrace{=}_{x_1\leftrightarrow x_4} v^{2\Delta_{\mathcal{O}}}\mathcal{G}\left(\frac{u}{v},\frac{1}{v}\right)$$

Conformal partial wave decomposition

$$\mathcal{G}(u,v) = \sum_{\mathcal{O}_{\Delta,\ell}}$$



$$\mathcal{G}(u,v) = \sum_{\mathcal{O}_{\Delta,\ell}} c^2_{\Delta,\ell}\, \underbrace{u^{\frac{\Delta-\ell}{2}} g_{\Delta,\ell}(u,v)}_{\text{conformal blocks}}$$

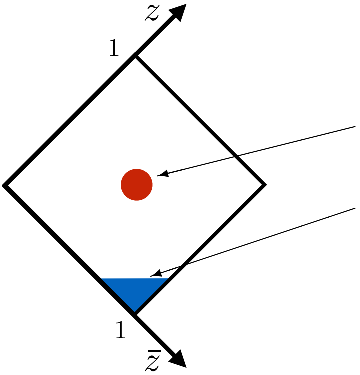
Let's combine the two...

Crossing or Bootstrap equation

A remarkable...but hard equation!

$$\underbrace{v^{\Delta_{\mathcal{O}}} \sum_{\Delta,\ell} c^2_{\Delta,\ell} u^{\frac{\Delta-\ell}{2}} g_{\Delta,\ell}(u,v)}_{\text{Easy to expand around } u=0,\, v=1} = \underbrace{u^{\Delta_{\mathcal{O}}} \sum_{\Delta,\ell} c^2_{\Delta,\ell} v^{\frac{\Delta-\ell}{2}} g_{\Delta,\ell}(v,u)}_{\text{Easy to expand around } u=1,\, v=0}$$

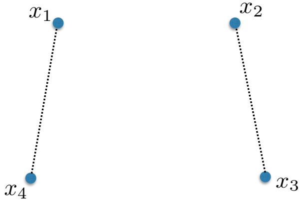
Study this equation in different regions, $u = z\bar{z}, v = (1-z)(1-\bar{z})$



- In the Euclidean regime $\bar{z} = z^*$.
- We can study crossing around $u = v = \frac{1}{4}$ (numerical bootstrap)
- In the Lorentzian regime $z, \bar{z}$ are independent real variables and we can consider $u, v \to 0$ (analytic bootstrap)

Analytic bootstrap

- In Minkowski space we can have $x_{23}^2 \to 0, x_{23} \neq 0$ (small v, any u)
- When some operators become null-separated the correlator develops singularities:



$$\longrightarrow \qquad \mathcal{G}(u,v) \sim \frac{h(u)}{v^{\alpha}}$$

- The whole correlator can be reconstructed from these singularities!

LSPT/Inversion formula [L.F.A.; Caron-Huot]

$$\mathcal{G}(u,v) = \int dudvK(u,v)sing[\mathcal{G}(u,v)] + \text{ambiguities}$$

- Hence the spectrum and OPE coef. of intermediate operators.

Generalised free fields

- Starting point - Generalised free fields (CFTs at $N = \infty$)

$$\mathcal{G}^{(0)}(u,v) = 1 + \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} + u^{\Delta_{\mathcal{O}}}$$

- Intermediate operators:

$$1 + \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} + u^{\Delta_{\mathcal{O}}} = 1 + \sum_{\Delta,\ell} a^{(0)}_{n,\ell} u^{(\Delta_{n,\ell}-\ell)/2} g_{n,\ell}(u,v)$$

Identity operator	Double trace operators
$\mathbb{I}$	$\mathcal{O}\Box^n \partial_{\mu_1}\cdots\partial_{\mu_\ell}\mathcal{O}$
$\Delta = \ell = 0$	$\Delta_{n,\ell} = 2\Delta_{\mathcal{O}} + 2n + \ell$

Example: Generalised free fields

- Note the answer diverges as $v \to 0$:

$$\mathcal{G}^{(0)}(u,v) \sim \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}}$$

- This singularity arises (via crossing) from the presence of the identity in the crossed channel:

$$\mathcal{G}^{(0)}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \mathcal{G}^{(0)}(v,u) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} (1 + \cdots)$$

- We could have reconstructed the whole correlator from this!
- Use this idea to compute $1/N^2$ corrections to GFF.

$$\mathcal{G}(u,v) = \mathcal{G}^{(0)}(u,v) + \frac{1}{N^2}\mathcal{G}^{(1)}(u,v) + \frac{1}{N^4}\mathcal{G}^{(2)}(u,v) + \cdots$$

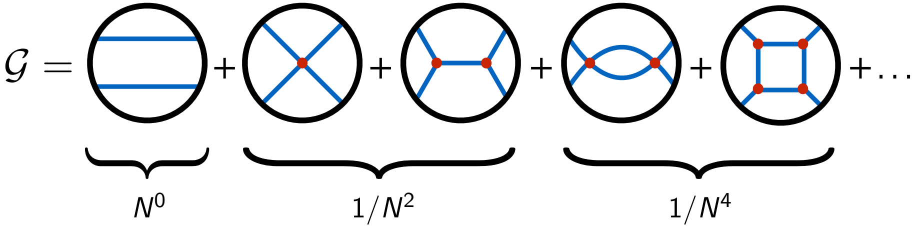
How does this map to AdS?

Large N CFTs

AdS/CFT

Large N CFT in D-dimensions (GFF + corrections) $\Leftrightarrow$ Gravitational theory in AdS_{D+1}

$$\frac{1}{N^2} \text{ expansion in CFT} \leftrightarrow \text{genus/loops in } AdS$$



- Direct computations in AdS are hard...Let's use CFT!

Large N holographic CFTs

$$\mathcal{G}(u,v) = \mathcal{G}^{(0)}(u,v) + \boxed{\frac{1}{N^2}\mathcal{G}^{(1)}(u,v)} + \frac{1}{N^4}\mathcal{G}^{(2)}(u,v) + \cdots$$

Two Sources of corrections

1. Double trace operators will acquire corrections:

$$\begin{aligned} \Delta_{n,\ell} - \ell &= 2\Delta_{\mathcal{O}} + 2n + \frac{1}{N^2}\gamma^{(1)}_{n,\ell} + \cdots \\ a_{n,\ell} &= a^{(0)}_{n,\ell} + \frac{1}{N^2}a^{(1)}_{n,\ell} + \cdots \end{aligned}$$

2. We can also have new intermediate operators at order $1/N^2$.

Which corrections are consistent with crossing symmetry and the structure of null singularities?

Large N holographic CFTs

Exchange solutions

In holographic CFT's we have the stress tensor (dual to the graviton)

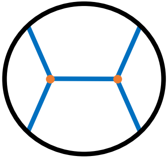
$$\mathcal{O} \times \mathcal{O} = 1 + [\mathcal{O}, \mathcal{O}]_{n,\ell} + \frac{1}{N^2} T_{\mu\nu}$$

- $T_{\mu\nu}$ in the crossed channel produces a precise divergence in the null limit:

$$\mathcal{G}^{(1)}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \mathcal{G}^{(1)}(v,u) \supset \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} v^{\frac{d-2}{2}} g_T(v,u)$$

- From this divergence we can fix $\gamma^{(1)}_{n,\ell}, a^{(1)}_{n,\ell}$!
- This solution corresponds to AdS graviton exchange

$$\mathcal{G}^{(1)}_{grav}(u,v) \sim$$

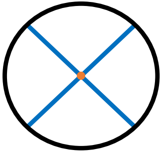


Truncated solutions

- In addition homogenous solutions to crossing with no extra null divergence (corresponding to the ambiguities)
- 'Truncated' $\gamma^{(1)}_{n,\ell}, a^{(1)}_{n,\ell}$ with finite support in the spin.
- Truncated solutions $\leftrightarrow$ local interactions in the *AdS* bulk.

  [Heemskerk, Penedones, Polchinski, Sully]

$$\mathcal{G}^{(1)}_{trunc}(u,v) \sim$$



Large N holographic CFTs

$$\mathcal{G}(u,v)=\mathcal{G}^{(0)}(u,v)+\frac{1}{N^2}\mathcal{G}^{(1)}(u,v)+\boxed{\frac{1}{N^4}\mathcal{G}^{(2)}(u,v)}+\cdots$$

- To order $1/N^4$ null divergences arise from the square of anomalous dimensions:

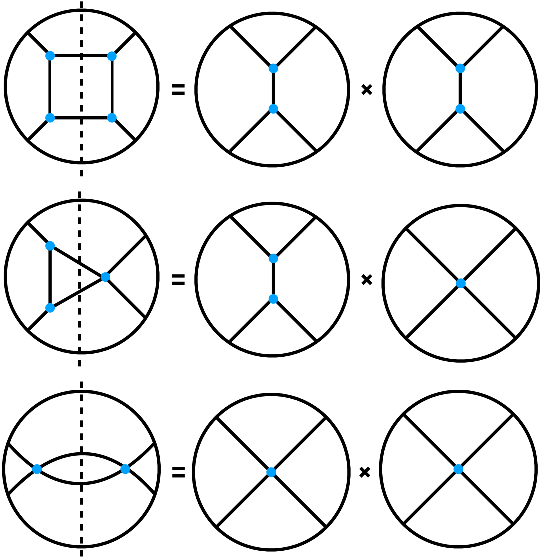
$$div\left[\left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}}\sum_{n,\ell} v_{n,\ell}^{\Delta_{\mathcal{O}}+\frac{1}{N^2}\gamma^{(1)}} g_{n,\ell}(v,u)\right]\sim\frac{\log^2 v}{N^4}\sum_{n,\ell}\left(\gamma_{n,\ell}^{(1)}\right)^2 g_{n,\ell}(v,u)$$

- Given $\left(\gamma_{n,\ell}^{(1)}\right)^2$ we can reconstruct $\mathcal{G}^{(2)}(u,v)$!

$$\left(\gamma^{(1)}\right)^2=\left(\gamma^{(1)}_{grav}+\gamma^{(1)}_{trunc}\right)^2=\gamma^{(1)}_{grav}\times\gamma^{(1)}_{grav}+\gamma^{(1)}_{grav}\times\gamma^{(1)}_{trunc}+\gamma^{(1)}_{trunc}\times\gamma^{(1)}_{trunc}$$

leads to the following contributions

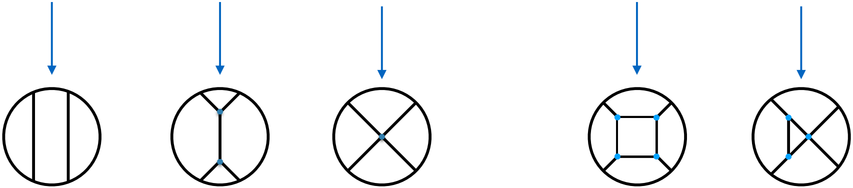
Contributions to order $1/N^4$



Double expansion in $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM falls into this category! with the following structure

$$\mathcal{G}^{(0)}(u,v) + \frac{1}{N^2}\left(\mathcal{G}^{(1)}_{grav}(u,v) + \frac{1}{\lambda^{3/2}}\mathcal{G}^{(1)}_{st}(u,v) + \cdots\right) + \frac{1}{N^4}\left(\mathcal{G}^{(2)}_{grav}(u,v) + \frac{1}{\lambda^{3/2}}\mathcal{G}^{(2)}_{st}(u,v) + \cdots\right)$$



We can compute all these contributions!

Mellin space [Mack; Penedones]

The string amplitude in $AdS_5 \times S^5$ is $M(s,t,u)$

$$\mathcal{G}(U,V) = \int_{-i\infty}^{i\infty} ds dt U^s V^t M(s,t,u) \Gamma^2(2-s)\Gamma^2(2-t)\Gamma^2(2-u)$$

where $s+t+u=2$

- Crossing symmetry $\rightarrow M(s,t,u)$ is completely symmetric.
- Gravity solution $\rightarrow M(s,t,u)$ a meromorphic function:

$$M^{(1)}_{sugra}(s,t,u) = \frac{1}{(s-1)(t-1)(u-1)}$$

- Truncated solutions/stringy corrections $\rightarrow M(s,t,u)$ is a polynomial. Nice basis

$$\sigma_2 = s^2+t^2+u^2, \qquad \sigma_3 = s^3+t^3+u^3$$

Mellin space

Genus zero expansion:

$$M^{genus\ 0}(s,t,u) = \frac{1}{(s-1)(t-1)(u-1)} + \alpha'^3 a + \alpha'^5 (b\sigma_2 + b_1) + \alpha'^6 (c\sigma_3 + c_1\sigma_2 + c_2) + \cdots$$

$gravity$ ↓ $\mathcal{R}^4$ ↓ $\partial^4\mathcal{R}^4$ ↓ $\partial^6\mathcal{R}^4$ ↓

Flat space limit: $s,t,u \to \infty, \alpha' \to 0$ with $s\alpha', t\alpha', u\alpha' \sim fixed$

Genus 0 flat space limit

$$M^{genus\ 0}(s,t,u) \to \frac{1}{stu} + \alpha'^3 a + \alpha'^5 b\sigma_2 + \alpha'^6 c\sigma_3 = \text{VS amplitude}$$

- b_1, c_1, c_2 are curvature corrections [recently fixed in a beautiful paper by Binder, Chester, Pufu, and Wang]

Mellin space and flat space limit

Genus-one string amplitude on $AdS_5 \times S^5$ in a α' expansion

$$M^{genus\ 1}_{grav.}(s,t,u) + \alpha'^3 M^{genus\ 1}_{st,1}(s,t,u) + \alpha'^5 M^{genus\ 1}_{st,2}(s,t,u) + \cdots$$

$$M^{genus\ 1}_{grav.}(s,t,u) = \sum_{m,n=2} \left(\frac{c_{mn}}{(s-2m)(t-2n)} + \text{crossed} \right)$$

$$M^{genus\ 1}_{st,1}(s,t,u) = P(s,t,u)\psi_0\left(2 - \frac{s}{2}\right) + \text{crossed}$$

- In the flat space limit they simplify a lot!

$$\begin{aligned} M^{genus\ 1}_{grav.}(s,t,u) &\rightarrow \text{Box function in 10D} \\ M^{genus\ 1}_{st,1}(s,t,u) &\rightarrow s^4 \log s + t^4 \log t + u^4 \log u \\ M^{genus\ 1}_{st,1}(s,t,u) &\rightarrow (87 s^6 + s^4 (t-u)^2) \log s + \cdots \end{aligned}$$

- Agrees exactly with the known results! **[Green, Russo, Vanhove]**

Conclusions

- We have presented the first genus one computation of a string theory amplitude in curved space-time.
- Together with the paper by Binder at. al. this provides the first genus one precision test of AdS/CFT.
- We can answer detailed questions, such as the structure of UV divergences, and in the flat space limit we recover the known structure.
- Can we now start asking quantitative questions about quantum gravity/string theory on curved spaces?
  - What is the space of modular functions that can appear?
  - What can we say about vertex operators in curved space?