# Building VOAs out of Higgs Branches

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In a superficial sense, this gives a geometric characterization of the associated VOA – it is a *chiral quantization* of the Higgs branch.

However, this is not a particularly constructive notion. It is just a fancy term to describe precisely the situation as outlined.

Question:

Is there a stronger sense in which these VOAs are encoded in the physics of their respective Higgs branches?

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Free Field Conjecture [CB, Meneghelli, Rastelli]

The VOA associated to an $\mathcal{N}=2$ SCFT admits a "free field realization" in terms of:

- A lattice VOA $\mathbb{V}\Pi_{d,d}$ for a lattice $\Pi$ of signature $(d,d)$ with $d=\dim_{\mathbb{H}}\mathcal{M}_H$.
- A $C_2$-cofinite VOA $\mathbb{V}[\mathcal{T}_{\text{IR}}]$, where $\mathcal{T}_{\text{IR}}$ is the infrared SCFT supported at a *generic* point on the Higgs branch (a point of maximal Higgsing).

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The particular form of the free field construction encodes transparently the structure of the Higgs branch as a holomorphic-symplectic variety.

# Plan for the talk

- Review of VOA/SCFT correspondence
- Rudiments of free field realizations and a key example
- Rank-one exceptional series: minimal nilpotent orbits
- Rank-two exceptional series: two-instanton moduli spaces
- Comments and open questions

# VOA/SCFT correspondence

[CB, Madalena Lemos, Pedro Liendo, Peelaers, Rastelli, Balt van Rees]

Long ago in 2013 we gave a construction of a vertex operator algebra given an $\mathcal{N}=2$ superconformal field theory in four dimensions.

$$\text{4d } \mathcal{N}=2 \text{ SCFT } \mathcal{T} \qquad \xrightarrow{\qquad\quad \mathbb{V} \qquad\quad} \qquad \text{VOA } \mathbb{V}[\mathcal{T}]$$

This is a concrete construction that takes place within the OPE algebra of the parent SCFT by way of a carefully chosen cohomological reduction.

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Consequently, a huge amount of detailed algebraic information about the full SCFT is encoded in the associated VOA, but it is often a challenge to extract it!

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Start with $\mathcal{N} = 2$ SCFTs *qua* OPE algebras,

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \sim \sum_k \frac{{c_{12}}^k \mathcal{O}_k(x_2)}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_k}} \;, \qquad x_{1,2} \in \mathbb{R}^4 \;.$$

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Local operators $\{\mathcal{O}_i(x)\}$ organized in representations of $\mathfrak{su}(2,2|2)$ superconformal algebra, with bosonic subalgebra

$$\mathfrak{su}(2,2)\times\mathfrak{su}(2)_R\times\mathfrak{u}(1)_r$$

along with sixteen fermionic symmetries,

- Poincaré supercharges: $\mathcal{Q}^I_\alpha$ and $\widetilde{\mathcal{Q}}^I_{\dot\alpha}$ with $I=1,2$, $\alpha=\pm$, $\dot\alpha=\pm$.
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We label operators by their charges $(E,j_1,j_2,R,r)$ under the Cartan subalgebra.

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More immediately tractable algebras can be extracted by thinking of the OPE algebra as a *dg-OPE algebra* with respect to a nilpotent element of $\mathfrak{su}(2,2|2)$ and passing to the cohomology.

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The last of these actually has the structure of a commutative (Poisson) vertex algebra, and arises in the *holomorphic-topological twist* of $\mathcal{N} = 2$ SCFTs. [Kapustin 2006]

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- Arises upon taking cohomology of mixed supercharge,

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For my purposes, a VOA structure on $\mathcal{V}$ amounts to an (associative) meromorphic OPE algebra in two dimensions.

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_k \frac{{c_{12}}^k \mathcal{O}_k(w)}{(z-w)^{h_1+h_2-h_k}} \ .$$

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Each operator can be expanded in a Laurent series,

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The *normally ordered product* of two operators is defined at the origin according to

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For future convenience, introduce secondary bracket which captures simple pole in OPE,

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- Null states are removed (*i.e.*, always in simple quotient) .

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- By further restricting to subspaces with $h = R$ or $h = R + r$, we recover $\mathcal{R}_H$ and $\mathcal{R}_{HL}$ (as Poisson algebras).

# Higgs branches from VOAs

[CB, Rastelli (2017)]

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We have conjectured an alternative method to extract $\mathcal{R}_H$ canonically from the VOA.

$$\begin{aligned} C_2(\mathcal{V}) &:= \operatorname{span}\{a_{-h_a-1}b\ , \quad a\ ,b \in \mathcal{V}\} \\ \mathcal{R}_{\mathcal{V}} &:= (\mathcal{V}/C_2(\mathcal{V})\ ,\ \mathrm{NO}(\ ,\ )\ ,\ \{\cdot,\cdot\}\ )\ . \end{aligned}$$

Here we are essentially "removing derivatives" from the VOA. The resulting $\mathcal{R}_{\mathcal{V}}$ is a commutative Poisson algebra by construction: *Zhu's commutative algebra*

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Higgs Branch Conjecture [CB, Rastelli]

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VOAs whose associated varieties are symplectic dubbed *quasi-Lisse* by T. Arakawa and K. Kawasetsu. Strong constraint (*e.g.*, modularity of characters of ordinary modules).

# VOAs from Higgs branches

# Free field realizations

[CB, Meneghelli, Rastelli (2019)]

It is often useful to realize a (potentially complicated) VOA as a vertex operator sub-algebra of a simpler VOA, such as a lattice VOA or some collection of $(\beta, \gamma)$ or free fermion VOAs. Recall a couple of famous examples.

# Free field realizations

[CB, Meneghelli, Rastelli (2019)]

It is often useful to realize a (potentially complicated) VOA as a vertex operator sub-algebra of a simpler VOA, such as a lattice VOA or some collection of $(\beta, \gamma)$ or free fermion VOAs. Recall a couple of famous examples.

## Virasoro from chiral boson (Feigin-Fuchs)

Given a chiral boson VOA

$$\varphi(z)\varphi(w) \sim \log(z-w)$$

Realize Virasoro algebra of general central charge $c = 1 - 12\alpha^2$ via background charge method,

$$T(z) = \frac{1}{2}(\varphi')^2 + \alpha\varphi'' \ .$$

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Affine $\mathfrak{sl}(2)$ currents at level $k$ given as follows,

$$\begin{aligned} J^+(z) &= \beta(z) \ , \\ J^3(z) &= -2(\beta\gamma) - \sqrt{2(k+2)}\varphi' \ , \\ J^-(z) &= -(\gamma\gamma\beta) - \sqrt{2(k+2)}(\gamma\varphi') + k\gamma' \ . \end{aligned}$$

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Remark: For any simple $\mathfrak{g}$, this is generalized to a construction involving $r_{\mathfrak{g}}$ chiral bosons and $\frac{1}{2}(d_{\mathfrak{g}} - r_{\mathfrak{g}})$ $(\beta, \gamma)$ pairs.

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[Critical-level AKM VOAs arise via the SCFT/VOA correspondence when considering the OPE algebra supported on real co-dimension two defects in six-dimensional $(2, 0)$ theories.]

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The associated variety for this VOA is the nilpotent cone of $\mathfrak{sl}(2)$,

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Singular vector at level $h = 3$ of the form $(J^A T^{\text{Sug}} + \ldots)$ generates all nulls.

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Adamovic constructed the *simple AKM VOA* at this level in terms of a lattice VOA $\mathbb{V}\Pi_{1,1}$:

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The free field realization is an *affinization* of this realization of $\mathbb{C}[\mathbb{C}^2/\mathbb{Z}_2]$ in $\mathbb{C}[T^*\mathbb{C}^\times]$. Here we are making replacements

$$X \longleftrightarrow e^{\delta+\varphi} \ , \quad Z \longleftrightarrow \frac{k}{2}(\varphi' - \delta') + \dots \ , \quad Y \longleftrightarrow \frac{-k^2}{4}\left(\frac{\varphi' - \delta'}{2}\right)^2 e^{-\delta-\varphi} + \dots$$

# A free field $R$-filtration

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An isotropic subalgebra $\mathrm{ISO}\mathbb{V}\Pi_{d,d} \subset \mathbb{V}\Pi_{d,d}$ admits a natural (good) filtration. We will identify with the physical $R$-filtration coming from four dimensions, and use it to introduce a formal $\hbar$ to organize "quantum" corrections.

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Define a monomial basis for $\mathrm{ISO}(\mathbb{V}\Pi_{d,d})$ using free field normal ordering. Assign $\tilde{R}$-grading according to

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Take the associated filtration of this grading as the $R$-filtration, quantum corrections are subleading in filtration.

# Rank one Deligne SCFTs

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Their associated VOAs are $V_{\frac{-h^\vee-6}{6}}(\mathfrak{g})$ for $\mathfrak{g}\in\{\mathfrak{a}_0,\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{d}_4,\mathfrak{e}_6,\mathfrak{e}_7,\mathfrak{e}_8\}$ .

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## Remarks

- $\mathfrak{a}_0$ is a formal entry in this list, it corresponds to the $\mathrm{Vir}_{(2,5)}$ VOA.
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The associated variety/Higgs branch is $\mathbb{O}_{\min}(\mathfrak{g})$ (we formally set $\mathbb{O}_{\min}(\mathfrak{a}_0) = \{\mathrm{pt.}\}$).

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- An isotropic subalgebra of the Lorentzian lattice VOA $\mathbb{V}\Pi_{1,1}$:

$$\bigoplus_{n=-\infty}^{\infty} \left(V_{\partial\varphi} \otimes V_{\partial\delta}\right) e^{\frac{n}{2}(\delta+\varphi)} .$$

- $2h^\vee - 4$ symplectic bosons $\{\xi_A\}$ associated to the symplectic vector space $\mathfrak{R}$.

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Can construct $\mathbb{C}[\mathbb{O}_{\min}(\mathfrak{g})]$ as Poisson subalgebra of functions on this cotangent bundle as before and affinize.

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Remark: These constructions are *super economical!* 58 bosons for $\mathfrak{e}_8$, compared to $248$ (or $240$ at critical level) from W-FF.

# Rank two Deligne SCFTs

[CB, Meneghelli, Peelaers, Rastelli (to appear)]

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We will make use a small number of facts about these theories:

- Their Higgs branches are the (centered) *two-$\mathfrak{g}$-instanton* moduli spaces on $\mathbb{C}^2$. Correspondingly, their Higgs chiral rings are generated by moment maps for $\mathfrak{su}(2) \times \mathfrak{g}$ global symmetry, along with $R = 3/2$ generators $\omega$ in the $(\mathbf{2}, \mathbf{Adj})$.

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This time (for $\mathfrak{g} \notin \{\mathfrak{g}_2, \mathfrak{f}_4\}$) these are the theories arising from *a pair* of D3 branes probing the same class of F-theory singularities as in the previous examples.

We will make use a small number of facts about these theories:

- Their Higgs branches are the (centered) *two-$\mathfrak{g}$-instanton* moduli spaces on $\mathbb{C}^2$. Correspondingly, their Higgs chiral rings are generated by moment maps for $\mathfrak{su}(2) \times \mathfrak{g}$ global symmetry, along with $R = 3/2$ generators $\omega$ in the $(\mathbf{2}, \mathbf{Adj})$.
- They are completely Higgsable, so no residual degrees of freedom.

# Rank two Deligne SCFTs

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- They are completely Higgsable, so no residual degrees of freedom.
- Central charges and flavour symmetry levels computed by Aharony and Tachikawa.

# Rank two Deligne SCFTs

[CB, Meneghelli, Peelaers, Rastelli (to appear)]

| $\mathfrak{g}$ | $h^\vee$ | $k_{2d}^{\mathfrak{g}}$ | $k_{2d}^{\mathfrak{su}(2)}$ | $c_{2d}$ | $\tilde{h}_{\min}$ | $a_{4d}$ | $r_i$ |
|---|---|---|---|---|---|---|---|
| $\mathfrak{g}(h^\vee)$ | $h^\vee$ | $\frac{-h^\vee-6}{3}$ | $\frac{-h^\vee-9}{6}$ | $-11-5h^\vee$ | $-\frac{9+9h^\vee}{24}$ | $\frac{23+8h^\vee}{24}$ | $\frac{h^\vee+6}{6}, \frac{h^\vee+6}{3}$ |
| $\mathfrak{a}_0$ | $\frac{6}{5}$ | | $-\frac{17}{10}$ | $-17$ | $-\frac{1}{5}$ | $\frac{163}{120}$ | $\frac{6}{5}, \frac{12}{5}$ |
| $\mathfrak{a}_1$ | $2$ | $-\frac{8}{3}$ | $-\frac{11}{6}$ | $-21$ | $-\frac{1}{3}$ | $\frac{13}{8}$ | $\frac{4}{3}, \frac{8}{3}$ |
| $\mathfrak{a}_2$ | $3$ | $-3$ | $-2$ | $-26$ | $-\frac{1}{2}$ | $\frac{47}{24}$ | $\frac{3}{2}, 3$ |
| $\mathfrak{g}_2$ | $4$ | $-\frac{10}{3}$ | $-\frac{13}{6}$ | $-31$ | $-\frac{2}{3}$ | $\frac{55}{24}$ | $\frac{5}{3}, \frac{10}{3}$ |
| $\mathfrak{d}_4$ | $6$ | $-4$ | $-\frac{5}{2}$ | $-41$ | $-1$ | $\frac{71}{24}$ | $2, 4$ |
| $\mathfrak{f}_4$ | $9$ | $-5$ | $-3$ | $-20$ | $-\frac{3}{2}$ | $\frac{95}{24}$ | $\frac{5}{2}, 5$ |
| $\mathfrak{e}_6$ | $12$ | $-6$ | $-\frac{7}{2}$ | $-71$ | $-2$ | $\frac{119}{24}$ | $3, 6$ |
| $\mathfrak{e}_7$ | $18$ | $-8$ | $-\frac{9}{2}$ | $-101$ | $-3$ | $\frac{167}{24}$ | $4, 8$ |
| $\mathfrak{e}_8$ | $30$ | $-12$ | $-\frac{13}{2}$ | $-161$ | $-5$ | $\frac{263}{24}$ | $6, 12$ |

Remark: Central charges satisfy $c_{2d} = c_{\text{Sug}}^{\mathfrak{g}} + c_{\text{Sug}}^{\mathfrak{su}(2)}$ except for $\mathfrak{g} = \mathfrak{su}(3)$, where both current algebras are at critical level!

# Rank two Deligne SCFTs

[CB, Meneghelli, Peelaers, Rastelli (to appear)]

Centred two-$\mathfrak{g}$-instanton moduli spaces for $\mathfrak{g} \in \{\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{d}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}$.

$$\begin{aligned}
\mu_{\mathfrak{g}}\,\omega\Big|_{(\frac{1}{2},\mathbf{1})} &= 0\,, & 4\mu^2_{\mathfrak{su}(2)}\Big|_{(0,\mathbf{1})} &= \mu^2_{\mathfrak{g}}\Big|_{(0,\mathbf{1})}\,,\\
\mu_{\mathfrak{g}}\,\omega\Big|_{(\frac{1}{2},\mathbf{Y}^*_{\mathbf{2}})} &= 0\,, & \mu_{\mathfrak{g}}\,\omega\Big|_{(\frac{1}{2},\mathbf{Adj})} &= 4\,\mu_{\mathfrak{su}(2)}\,\omega\Big|_{(\frac{1}{2},\mathbf{Adj})}\,,\\
\mu^3_{\mathfrak{g}}\Big|_{(0,\mathbf{X_2})} &= b_2\,\omega^2\Big|_{(0,\mathbf{X_2})}\,, & \omega^2\Big|_{(1,\mathbf{Y}^*_{\mathbf{2}})} &= -\mu_{\mathfrak{su}(2)}\,\mu^2_{\mathfrak{g}}\Big|_{(1,\mathbf{Y}^*_{\mathbf{2}})}\,,\\
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$\{\mathbf{X_k}, \mathbf{Y_k}, \mathbf{Y^*_k}\}$ uniform notations for representations in the Deligne series [Cohen, de Man]

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\mu_{\mathfrak{g}}^3\Big|_{(0,\mathbf{X_2})} &= b_2\, \omega^2\Big|_{(0,\mathbf{X_2})}\,, & \omega^2\Big|_{(1,\mathbf{Y_2^*})} &= -\mu_{\mathfrak{su}(2)}\, \mu_{\mathfrak{g}}^2\Big|_{(1,\mathbf{Y_2^*})}\,,\\
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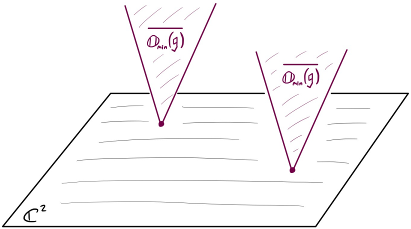
What a mess!

# Rank two Deligne SCFTs

[CB, Meneghelli, Peelaers, Rastelli (to appear)]

From the point of view of the F-theory construction, it is clear that this Higgs branch has the structure of two copies of the one-instanton moduli space, fibred over $\mathbb{C}^2/\mathbb{Z}_2$.



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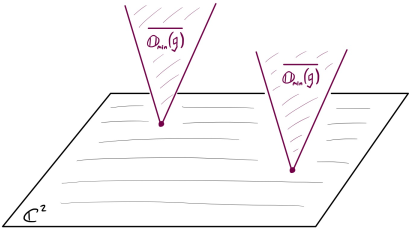


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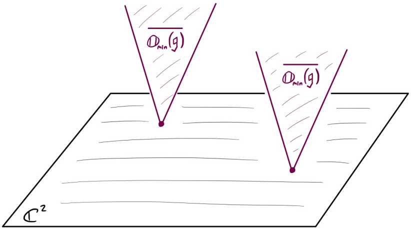


This will be the most instructive way to think about them.

Indeed, we adopt an intermediate point of view: try to build a free field realization in terms of the effective field theory on the locus where the residual theory is two copies of the rank-one SCFT!

# Rank two Deligne SCFTs

[CB, Meneghelli, Peelaers, Rastelli (to appear)]

So we will have a construction in terms of

- Two copies of the rank-one Deligne VOA $V_{\frac{-h^\vee-6}{6}}(\mathfrak{g})$:

$$\mathcal{J}_{1,2}^{A}(z)\ , \quad A = 1, \ldots, \dim\mathfrak{g}\ .$$

- Isotropic subalgebra of the Lorentzian lattice VOA $\mathbb{V}\Pi_{1,1}$:

$$\bigoplus_{n=-\infty}^{\infty} (V_{\partial\varphi} \otimes V_{\partial\delta})\, e^{\frac{n}{2}(\delta+\varphi)}\ .$$

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Can further express rank-one VOAs using the previous free field realization, since there all nulls vanish identically.

$$\dim_{\mathbb{H}} \widetilde{\mathcal{M}_{g,2}} = 2h^\vee - 1 = 2(h^\vee - 1) + 1$$

# Rank two Deligne SCFTs

[CB, Meneghelli, Peelaers, Rastelli (to appear)]

Free field realization:

$$\mathcal{J}^A(z) = \left(\mathcal{J}_1^A + \mathcal{J}_2^A\right) ,$$

$$j_{++}(z) = e^{\delta(z)+\varphi(z)} ,$$

$$j_{+-}(z) = \tfrac{k_{\mathfrak{su}(2)}}{2}\,\partial\varphi(z) ,$$

$$j_{--}(z) = \left(-S^{\natural} + \left(\left(\tfrac{k_{\mathfrak{su}(2)}}{2}\,\partial\delta\right)^2 - \tfrac{k_{\mathfrak{su}(2)}(k_{\mathfrak{su}(2)}+1)}{2}\partial^2\delta\right)\right)\left(e^{-(\delta+\varphi)}\right) .$$

$$S^{\natural} = (k_{\mathfrak{su}(2)}+2)\left(T_1^{\text{Sug}} + T_2^{\text{Sug}} - T_{12}^{\text{Sug}}\right) ,$$

$$\mathcal{W}_+^A(z) = \left(\mathcal{J}_1^A - \mathcal{J}_2^A\right) e^{\frac{1}{2}(\delta(z)+\varphi(z))} ,$$

$$\mathcal{W}_-^A(z) = \left(-\mathcal{U}^A(z) - (\mathcal{J}_1^A - \mathcal{J}_2^A)\tfrac{k_{\mathfrak{su}(2)}}{2}\,\partial\delta(z)\right)\left(e^{-\frac{1}{2}(\delta(z)+\varphi(z))}\right) ,$$

$$\mathcal{U}^A = \left(-\tfrac{4(2+k_{\mathfrak{su}(2)})}{k_{\mathfrak{g}}+h^{\vee}}\right)\tfrac{1}{2}\,if^A_{BC}\,\mathcal{J}_1^B\,\mathcal{J}_2^C + k^{\mathfrak{g}}\left(\tfrac{k_{\mathfrak{su}(2)}+2}{k_{\mathfrak{g}}+h^{\vee}}\right)\partial(\mathcal{J}_1^A - \mathcal{J}_2^A) ,$$

Form of generators completely fixed by filtration-compatible Ansatz and basic closure requirements.

# Rank two Deligne SCFTs

[CB, Meneghelli, Peelaers, Rastelli (to appear)]

We find a unique expression for the non-trivial $\mathcal{W} \times \mathcal{W}$ OPE such that algebra closes on Higgs branch generators (plus stress tensor in case of $\mathfrak{g} = \mathfrak{su}(3)$)

$$\begin{aligned}
\mathcal{W}^A_\alpha(z)\, \mathcal{W}^B_\beta(w) \sim\ & \frac{c_1\ \epsilon_{\alpha\beta}\ \kappa^{AB}}{(z-w)^3} + \frac{ic_2\ \epsilon_{\alpha\beta}\ f^{AB}{}_C\ \mathcal{J}^C(w) + c_3\ \kappa^{AB}\ j_{\alpha\beta}(w)}{(z-w)^2} \\
& + \frac{1}{z-w}\Big(ic_4\ \epsilon_{\alpha\beta}\ f^{AB}{}_C\ \partial\mathcal{J}^C(w) + c_5\ \kappa^{AB}\ \partial j_{\alpha\beta}(w) \\
& \qquad + c_6\ \kappa^{AB}\ \epsilon_{\alpha\beta}\ (jj)(w) + ic_7\ f^{AB}{}_C\ (j_{\alpha\beta}\mathcal{J}^C)(w) \\
& \qquad + \epsilon_{\alpha\beta}\Big(c_8\ \mathbb{1}^{(AB)}(w) + c_9\ \mathbb{Y}_2^{(AB)}(w) + c_{10}\ \mathbb{Y}_2^{*(AB)}(w)\Big)\Big)\ .
\end{aligned}$$

where coefficients take fixed form in terms of $h^\vee$:

$$c_1 = 1\,,\quad c_2 = -\frac{3}{6+h^\vee}\,,\quad c_3 = \frac{6}{9+h^\vee}\,,\quad c_4 = -\frac{3}{2(6+h^\vee)}\,,\quad c_5 = \frac{3}{9+h^\vee}\,,$$

$$c_6 = -\frac{9}{(-3+h^\vee)(9+h^\vee)}\,,\quad c_7 = -\frac{18}{(6+h^\vee)(9+h^\vee)}\,,\quad c_8 = -\frac{9h^\vee}{2(-3+h^\vee)(6+h^\vee)}\,,$$

$$c_9 = -\frac{9}{(6+h^\vee)(9+h^\vee)}\,,\quad c_{10} = \frac{3}{6+h^\vee}\,.$$

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These VOAs are now equipped with filtrations inherited from those of their free field spaces, which allows us to see some interesting structure that is obscure at the level of the unfiltered VOA.

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These sorts of delicate rearrangements in the associated graded are a *hallmark of the physical $R$-filtration*.

# Comments on irreducible pieces

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An important generalization comes from cases where the low-energy physics of the Higgs phase is not completely encoded in the geometry of the Higgs branch, but there are residual interacting degrees of freedom, *e.g.*,

- $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{g}$ supports $r_{\mathfrak{g}}$ free vector multiplets.
- Class $\mathcal{S}$ theories of type $\mathfrak{g}$ for genus $g \geqslant 1$ support $g \times r_{\mathfrak{g}}$ free vector multiplets.
- $(A_1, D_{2n+1})$ Argyres-Douglas supports $(A_1, A_{2n-2})$ theory.
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In these cases, the free field realization is equipped with an extra factor of the $C_2$ co-finite VOA associated to those residual degrees of freedom.

# Conclusions

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- For theories with sufficiently nice Higgs branches, our approach gives a powerful alternative to "bootstrapping" the associated VOA.
  Essentially all linear quiver SCFTs can be understood by a recursive application of some of the techniques mentioned here.
  Non-Lagrangian class $\mathcal{S}$ theories such as $T_4$ seem to be accessible as well. New viewpoint on so-called *Moore-Tachikawa symplectic varieties*.

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  Non-Lagrangian class $\mathcal{S}$ theories such as $T_4$ seem to be accessible as well. New viewpoint on so-called *Moore-Tachikawa symplectic varieties*.
- Examples I discussed all had "fibration over $\mathbb{C}^2/\mathbb{Z}_2$" in the geometry. Generalizes to other canonical singularities.
  Should also generalize to more general transverse slices to nilpotent orbits.

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  Conjecture that these should be highly constrained by four-dimensional unitarity.
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  Four-dimensional physics interpretation of screening charges?
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  Conjecture that these should be highly constrained by four-dimensional unitarity.
  More generally, unitarity constrains allowed Higgs branches.
- What role is four-dimensionality playing in this story?
  Some evidence of a similar for associated varieties/free field realizations beyond cases arising from (unitary) four dimensional physics.

Tack så mycket!