3d Quantum Modularity

Miranda C. N. Cheng
University of Amsterdam

Universiteit van Amsterdam

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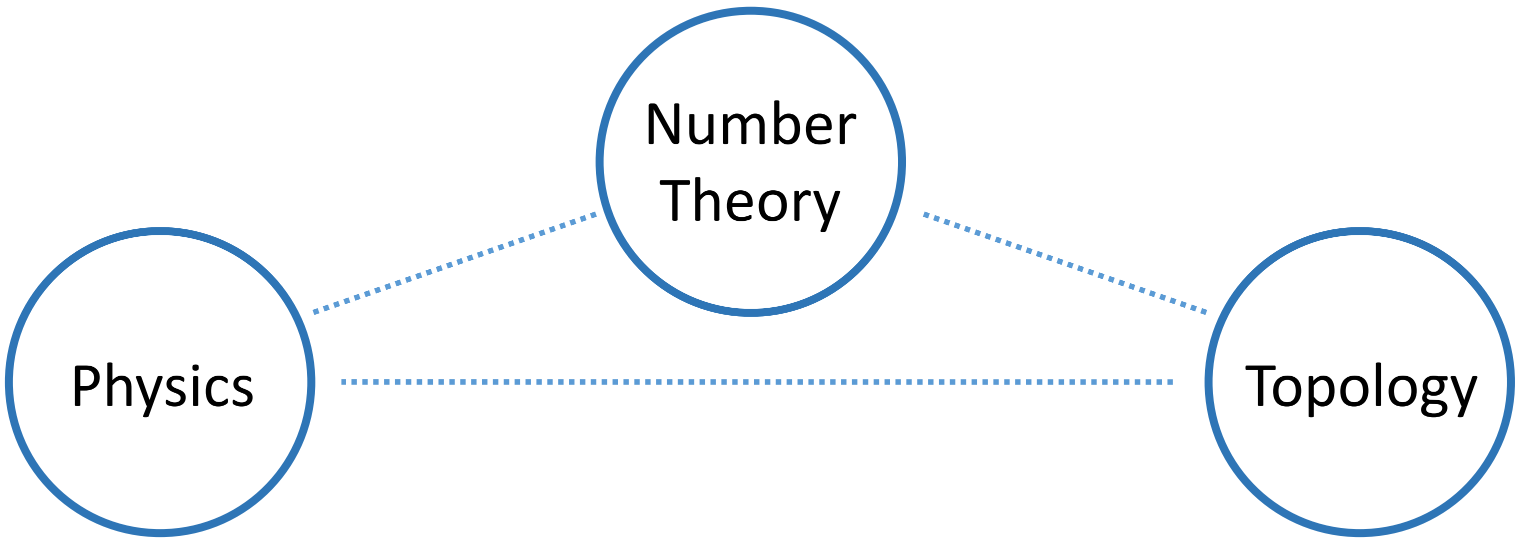
3-Manifold Inv.
$\widehat{Z}_a(M_3;\tau)$
Quantum
Modular Form



Motivations:

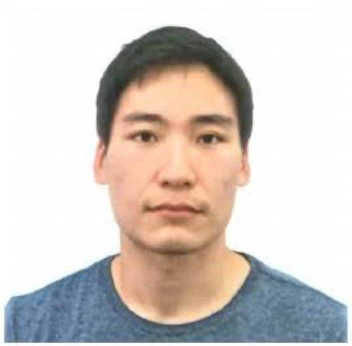
- q-invariants for (closed) 3-manifolds;
- natural structure beyond modular forms;
- M-theory, $3d$-$3d$, and $3d$ SQFT.

Based on:

- *3d Modularity*, arxiv:1809.10148
w. S. Chun, F. Ferrari, S. Gukov, S. Harrison

- *3d Modularity: Higher and Deeper*, to appear
w. S. Chun, B. Feigin, F. Ferrari, S. Gukov, S. Harrison

- *Three-Manifold Invariants and Indefinite Theta Functions*,
to appear, w. G. Sgroi






Outline

I. Background

II. The Connection

III. Beyond This Talk

I. Background

3-Manifold Inv.
$\widehat{Z}_a(M_3;\tau)$

Quantum
Modular Form

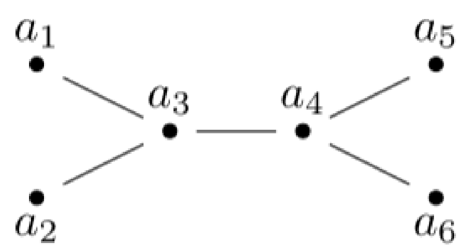
main ref. [Gukov-Pei-Putrov-Vafa '17]

$\widehat{Z}_a(M_3;\tau)$: Mathematical Definition

M_3: Plumbed 3-manifold, determined by its **plumbing graph** Γ.

weighted graph $\Gamma := (V, E, a)$, $a : V \to \mathbb{Z}$.

e.g.



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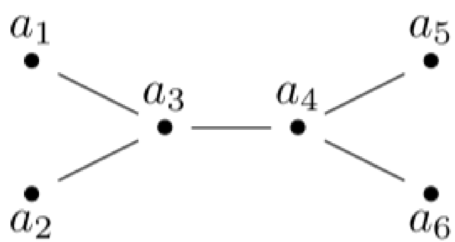

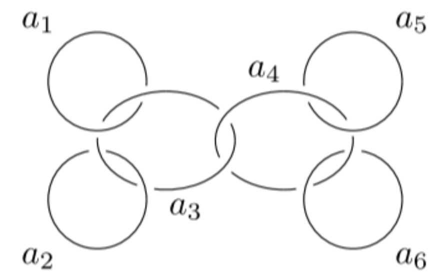

plumbing graph Γ ⟷ framed link L

↕

adjacency matrix M

$$M = \begin{pmatrix} a_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 1 & 1 & a_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_4 & 1 & 1 \\ 0 & 0 & 0 & 1 & a_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & a_6 \end{pmatrix}$$

$\widehat{Z}_a(M_3;\tau)$: Mathematical Definition

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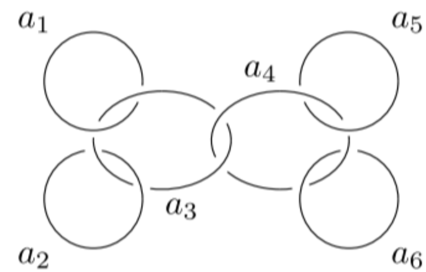


plumbing graph Γ ⟷ framed link L

↕ ↓

adjacency matrix M

plumbed $M_{3,\Gamma}$

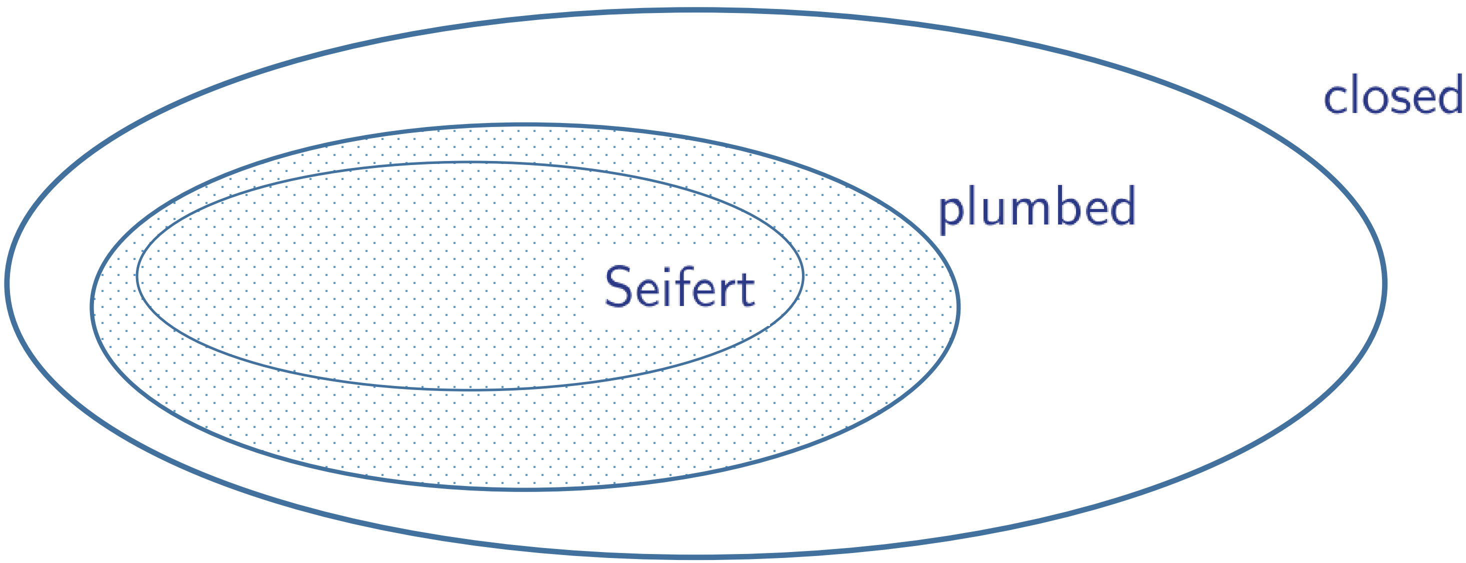
obtained with surgery along L

$$M = \begin{pmatrix} a_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 1 & 1 & a_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_4 & 1 & 1 \\ 0 & 0 & 0 & 1 & a_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & a_6 \end{pmatrix}$$

$H_1(M_{3,\Gamma};\mathbb{Z}) \cong \mathbb{Z}^{|V|}/M\mathbb{Z}^{|V|}$ (Coker M)

$\widehat{Z}_a(M_3;\tau)$: Mathematical Definition

M_3: Plumbed 3-manifold, determined by its **plumbing graph** Γ.



$\widehat{Z}_a(M_3;\tau)$: Mathematical Definition

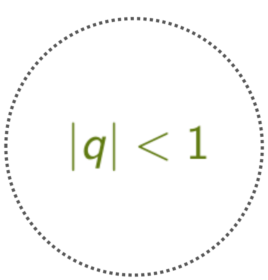
Def: For a weighted graph Γ with negative-definite M, and for a given $a \in (2\mathbb{Z}^{|V|}+\delta)/2M\mathbb{Z}^{|V|} \cong H_1(M_{3,\Gamma},\mathbb{Z})$, define the theta function

$$\Theta_a^M(\tau;\mathbf{z}) := \sum_{\ell \in a+2M\mathbb{Z}^{|V|}} q^{-(\ell, M^{-1}\ell)}\mathbf{z}^{\ell} \qquad (q = e^{2\pi i\tau})$$

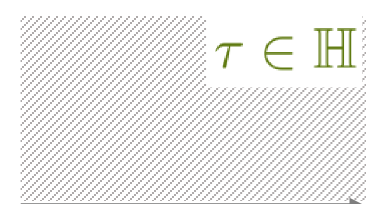
Rk:

1. $a \in \{\text{inequiv. SU(2) Ab. flat conn.}\} \cong \{\text{Spin}^c \text{ str. on } M_{3,\Gamma}\}$.

2. In this case $\Theta_a^M(\tau;\mathbf{z})$, and hence $\widehat{Z}_a(\tau)$, converges when $|q|<1 \Leftrightarrow \tau \in \mathbb{H}$.



unit disk	$\leftarrow$	upper-half plane $\mathbb{H}$
$q := e^{2\pi i\tau}$	$\leftarrow\!\!\mid$	$\tau = x+iy$



$\widehat{Z}_a(M_3;\tau)$: Mathematical Definition

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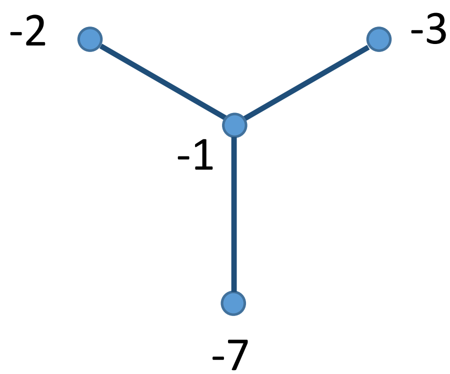
and

$$\boxed{\widehat{Z}_a(M_{3,\Gamma};\tau) := (\pm)\, q^{\Delta} \oint \prod_{v \in V} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta_a^M(\tau;\mathbf{z})}$$

Rk: 3. $q^c \widehat{Z}_a(\tau) \in \mathbb{Z}[[q]]$.

$\widehat{Z}_a(M_3;\tau)$: Mathematical Definition

eg.



$$M_{3,\Gamma} = \Sigma(2,3,7) = \{x^2 + y^3 + z^7 = 0\} \cap S^5$$

$$\widehat{Z}_0(\Sigma(2,3,7);\tau) = q^{\frac{1}{2}}(1 - q - q^5 + q^{10} - q^{11} + \dots)$$

$$= q^{\frac{1}{2}-\frac{1}{168}} \sum_{\substack{k\geq 0 \\ k^2\equiv 1\ (42)}} \left(\frac{k}{21}\right) q^{\frac{k^2}{168}}$$

$\pm 1, 0$

$\widehat{Z}_a(M_3;\tau)$: Physical Picture

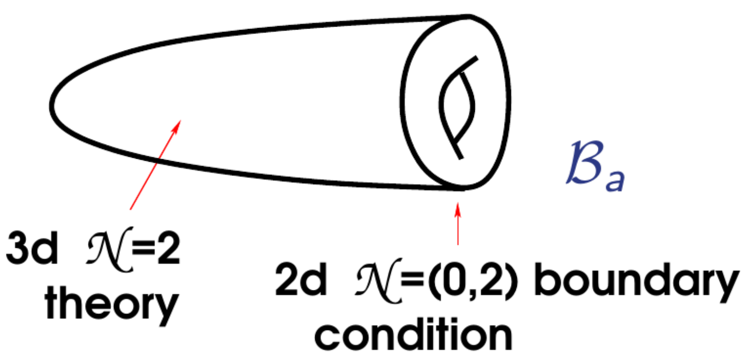
6d (2,0) $G(=SU(2))$-theory on M_3

3d $\mathcal{N}=2$ thy $\mathcal{T}_G[M_3]$

susy B.C. $\mathcal{B}_a$

M_3 Topology

Ab. G flat connections

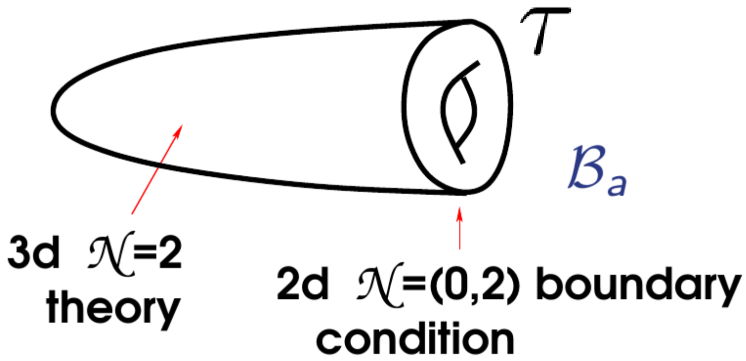


$$\widehat{Z}_a(\tau) \; := \; Z_{\mathcal{T}_G[M_3]}(D^2 \times_\tau S^1; \mathcal{B}_a)$$

$$= \text{“Half-Index”}$$

$$= \text{top. inv. of } M_3$$

$\widehat{Z}_a(M_3;\tau)$: Physical Picture



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$$= \text{“\textbf{Half-Index}”}$$

$$= \text{top. inv. of } M_3$$

3d bulk coupled to a 2d boundary CFT $\Rightarrow$

Some kind of residual **modularity** is expected if the bulk theory is “somewhat trivial”.

$\widehat{Z}_a(M_3;\tau)$ and Z_{CS}

$Z_{\mathrm{CS}}(M_3; k)$; $k \in \mathbb{Z}$ is the (effective) level.

Question: Can we go from $\mathbb{Z}$ to $\mathbb{H}$:
a q-series inv. for 3-man. extending Z_{CS}?

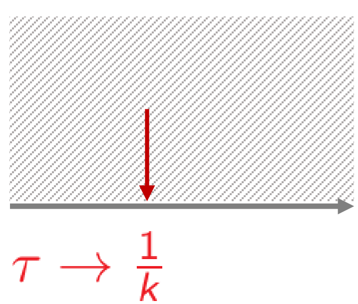
$\widehat{Z}_a(M_3;\tau)$ and Z_{CS}

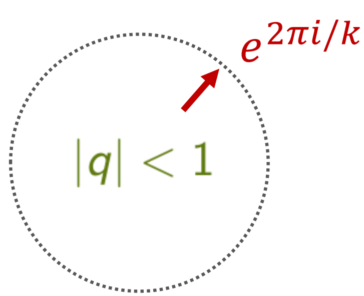
$Z_{\mathrm{CS}}(M_3;k)$; $k \in \mathbb{Z}$ is the (effective) level.

Question: Can we go from $\mathbb{Z}$ to $\mathbb{H}$:
a q-series inv. for 3-man. extending Z_{CS}?

Answer: $\widehat{Z}_a(\tau)$, related to Z_{CS} by $\widehat{Z}_a(\tau) \xrightarrow[\text{summed over } a]{\text{radial limit}} Z_{\mathrm{CS}}$

$$Z_{\mathrm{CS}}(M_3;k) = \sum_{a,b\in H_1(M_3,\mathbb{Z})} e^{2\pi i k\cdot \mathbf{lk}(a,a)} \left(\lim_{\tau\to\frac{1}{k}} S^{(A)}_{ab}\, \widehat{Z}_b(\tau) \right)$$



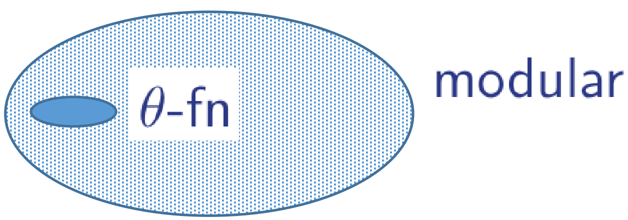


I. Background

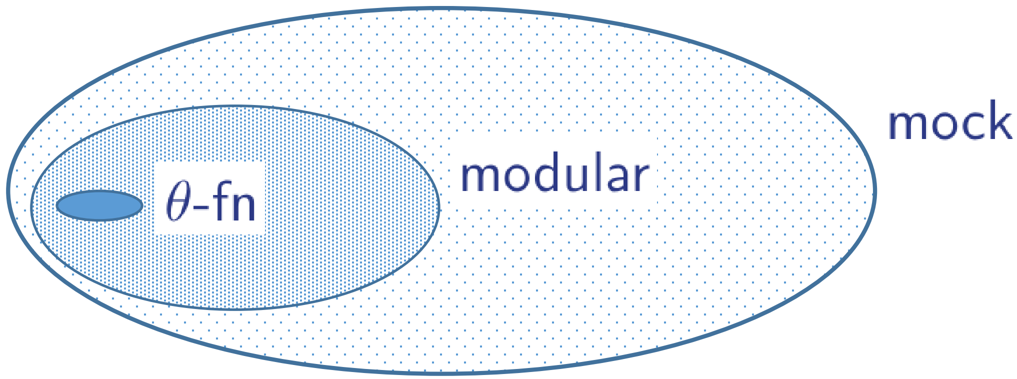
3-Manifold Inv.
$\widehat{Z}_a(M_3;\tau)$

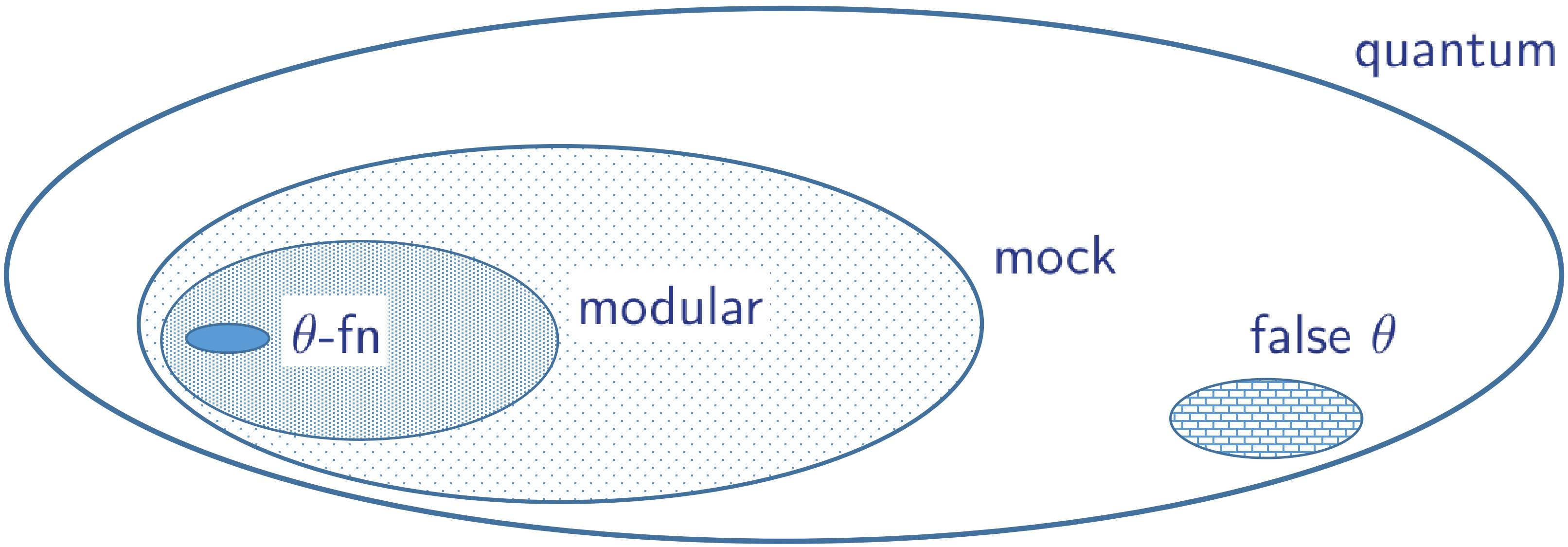
Quantum
Modular Form

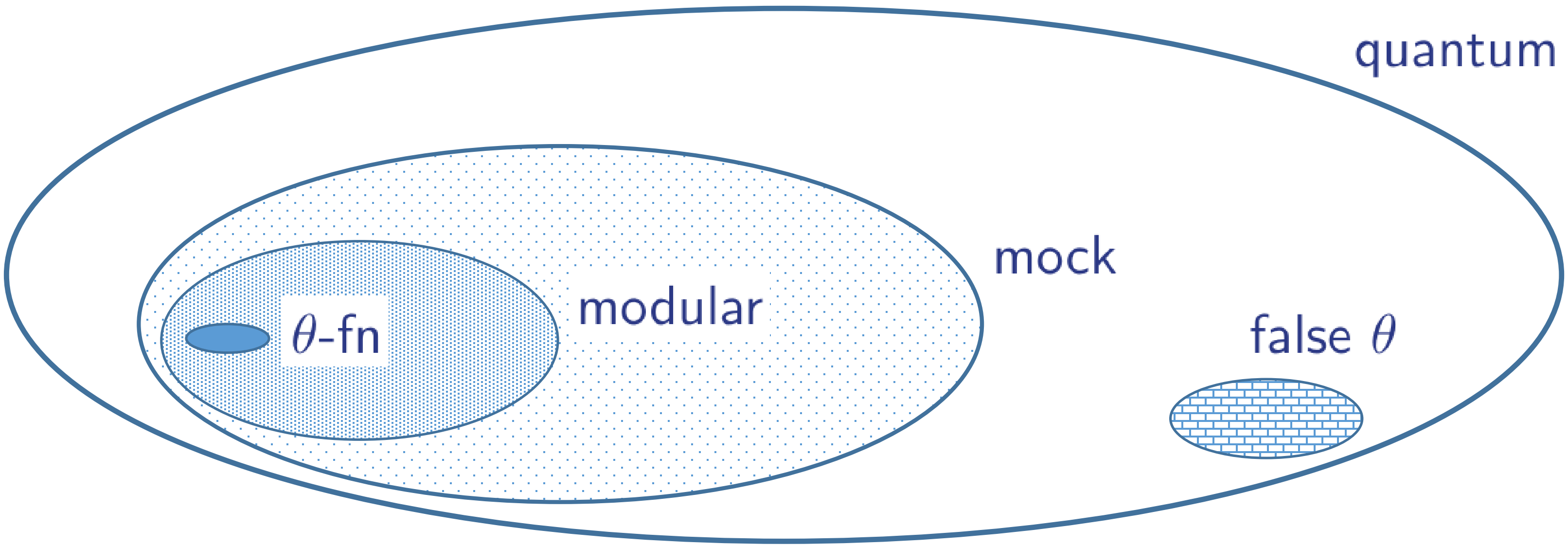
main ref. [Zagier '10]



θ-fn
modular


mock
modular
θ-fn


quantum
mock
modular
θ-fn
false θ



Examples: False Theta Functions, Mock Modular Forms,...

Applications: Kashaev invariants, log CFT characters, $\widehat{Z}_a(q)$, ...

Modular Forms

$\mathbb{H}$ ↺

Symmetry: $\tau \mapsto \gamma\tau := \dfrac{a\tau + b}{c\tau + d}$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \supset SL_2(\mathbb{Z}) \supseteq G$$

Modular Forms

Consider a holomorphic fn f on $\mathbb{H}$.

Def (modular transf. of weight k): $f|_k\gamma(\tau) := f(\gamma\tau)(c\tau + d)^{-k}$

Def (modular form of weight k, $f \in M_k(G)$): $f|_k\gamma = f \quad \forall\gamma \in G$

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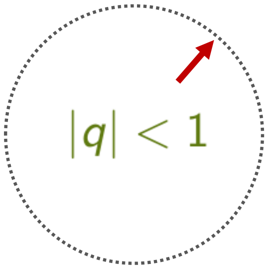
Example: Lattice θ-functions

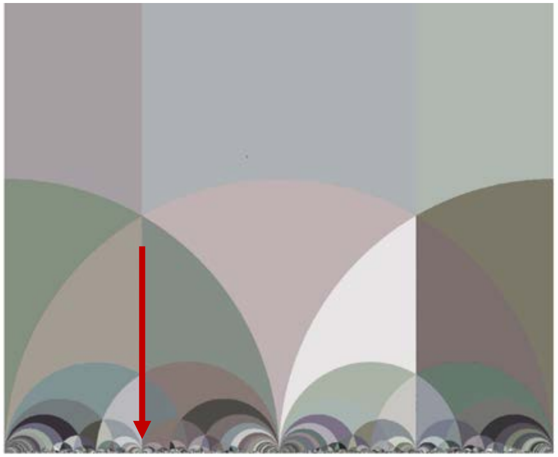
- $\Lambda = \mathbb{Z}$, $\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2} \in M_{1/2}$
- $\Lambda = \sqrt{2m}\mathbb{Z}$, $\Lambda^*/\Lambda \cong \mathbb{Z}/2m$,

$$\theta^0_{m,r}(\tau) = \sum_{k\equiv r \ (2m)} q^{\frac{k^2}{4m}} \in M_{1/2}$$

$$\theta^1_{m,r}(\tau) = \sum_{k\equiv r \ (2m)} kq^{\frac{k^2}{4m}} \in M_{3/2}$$

Quantum Modular Forms





$\tau \to \alpha \in \mathbb{Q}$

Consider $f : \mathbb{Q} \to \mathbb{C}$.

Def (Quantum Modular Form):

f is a QMF if the error of modularity $h_\gamma := f - f|_\gamma$ is a "nice" (*e.g. piecewise analytic*) function for all $\gamma \in G$.

False, Mock $\subset$ Quantum

Consider a modular form g of weight w.

Def (Eichler integrals): *

$$\tilde{g}(\tau) := \int_{\tau}^{i\infty} g(\tau')(\tau' - \tau)^{w-2} d\tau' \qquad \text{(holomorphic)}$$

$$g^*(\tau) := \int_{-\bar{\tau}}^{i\infty} g(\tau')(\tau' + \tau)^{w-2} d\tau' \qquad \text{(non-holomorphic)}$$

Rk: $\tilde{g} - \tilde{g}|_{\gamma}$ and $g^* - g^*|_{\gamma}$ are period integrals $\to$ quantum modularity.

$*$ some irrelavant constant factors ignored.

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Example: False θ-function

$$\widetilde{\theta^1_{m,r}}(\tau) = \sum_{\substack{k \in \mathbb{Z} \\ k \equiv r (2m)}} \mathrm{sgn}(k)\, q^{k^2/4m}$$

↘ false

$*$ some irrelavant constant factors ignored.

False, Mock $\subset$ Quantum

Consider a holomorphic fn f on $\mathbb{H}$.

Def (mock modular forms) [Zwegers '02]: f is a mmf if there exists a modular form $g = \mathrm{shad}(f)$ (the *shadow*) such that $\hat{f} := f - g^*$ satisfies $\hat{f} = \hat{f}|_\gamma \quad \forall\, \gamma \in G$.

Rk: $f - f|_\gamma = g^* - g^*|_\gamma$ is a period integral $\to$ quantum modularity.

Example : Ramanujan's Mock θ Functions

$$F_0(\tau) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-q^{n+k})} = 1 + q + q^3 + q^4 + O(q^5)$$

$$shad(F_0)(\tau) = \sum_{\substack{i \in \mathbb{Z}/42 \\ i^2 \equiv 1\ (42)}} \left(\frac{i}{21}\right) \theta^1_{42,i}(\tau)$$

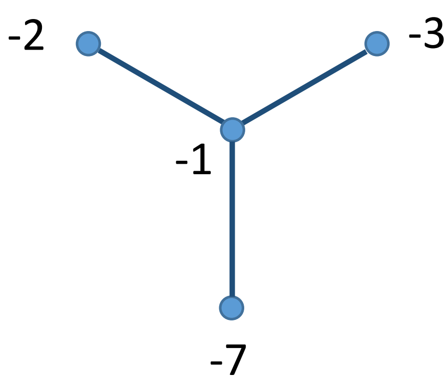
False, Mock $\subset$ Quantum

Example : Ramanujan's Mock θ Functions

$$F_0(\tau) = \sum_{n \geq 0} \frac{q^{n^2}}{(q^{n+1};q)_n} = 1 + q + q^3 + q^4 + O(q^5)$$

$$shad(F_0)(\tau) = \sum_{\substack{i \in \mathbb{Z}/42 \\ i^2 \equiv 1\ (42)}} \left(\frac{i}{21}\right) \theta^1_{42,i}(\tau)$$

eg.



$$\hat{Z}_0(\Sigma(2,3,7),\tau)$$

$$= q^{\frac{83}{168}} \sum_{\substack{k \geq 0 \\ k^2 \equiv 1\ (42)}} \left(\frac{k}{21}\right) q^{\frac{k^2}{168}}$$

$$= q^{\frac{83}{168}}\, \widetilde{shad(F_0)}(\tau) = q^{\frac{83}{168}} \sum_{\substack{i \in \mathbb{Z}/42 \\ i^2 \equiv 1\ (42)}} \left(\frac{i}{21}\right) \widetilde{\theta^1_{42,i}}(\tau)$$

II. 3d Quantum Modularity

3-Manifold Inv.
$\widehat{Z}_a(M_3;\tau)$

Quantum
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Applications:

Quantum modularity

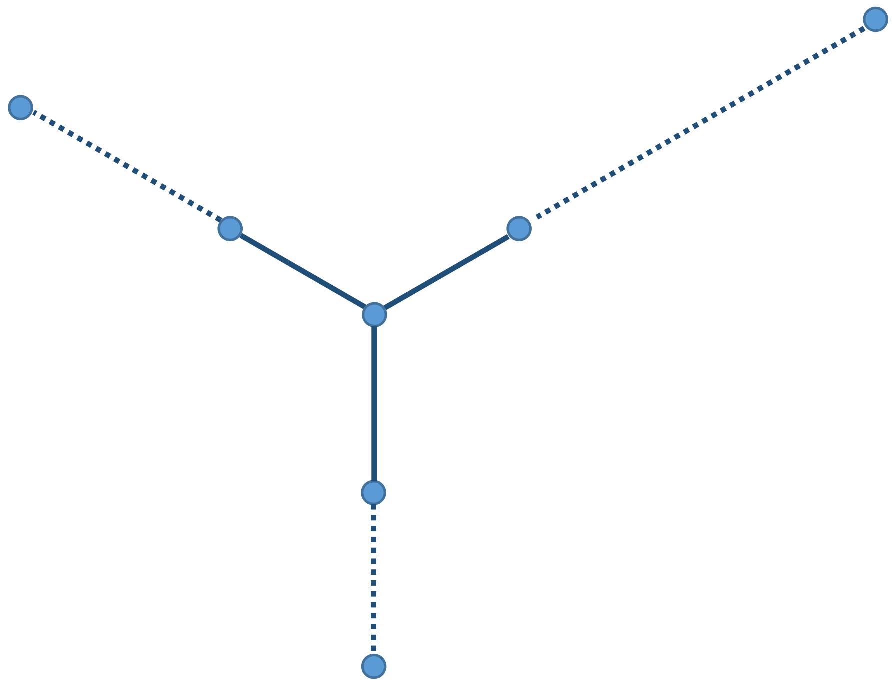
- helps to determine the q-invariants;
- leads to new ways of retrieving topological information;
- gives hints about the physical theories.

II.

See also important previous and ongoing work on a related topic (Kashaev invariants of knots):
Zagier '10, Garoufalidis-Zagier '13 and to appear, Dimofte-Garoufalidis '15, Hikami-Lovejoy '14, ….

In this talk we focus on the most tractable families of examples:

Γ = 3-pronged star



False 3-Manifolds

Theorem : Negative three-stars are false.

[MC-Chun-Ferrari–Gukov-Harrison, Bringmann-Mahlburg-Milas '18]

For any three-pronged star weighted graph Γ of *negative type*, the functions

$$\widehat{Z}_a(M_{3,\Gamma};\tau) := (\pm)\, q^{\Delta} \oint \prod_{v\in V} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta^M_a(\tau;\mathbf{z})$$

is a false theta function. In particular, there exists a unique $m = m(\Gamma) \in \mathbb{Z}_{>0}$ and a $c \in \mathbb{Q}$ such that

$$q^{-c}\widehat{Z}_a(\tau) \in \mathrm{span}_{\mathbb{Z}}\left\{\widetilde{\theta^1_{m,r}}, r \in \mathbb{Z}/2m\right\} \quad \forall\ a.$$

Very tractable!

See also earlier work by [Lawrence–Zagier '99] and Hikami.

A Puzzle

Upon flipping orientation, we have

$$Z_{\mathrm{CS}}(-M_3; k) = Z_{\mathrm{CS}}(M_3; -k)$$

Recall

$$\tau \to \frac{1}{k}$$

$$\widehat{Z}_a(\tau) \xrightarrow[\text{summed over } a]{\text{radial limit}} Z_{\mathrm{CS}}$$

A Puzzle

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Recall

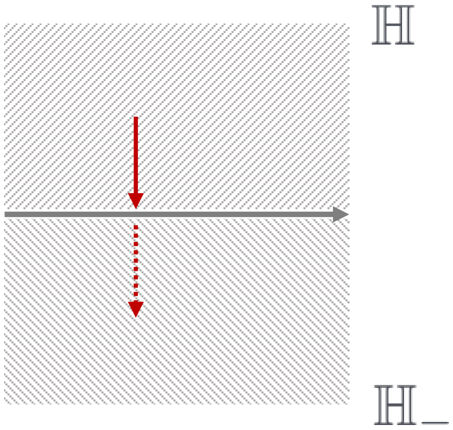
$$\widehat{Z}_a(\tau) \xrightarrow[\text{summed over } a]{\text{radial limit}} Z_{\mathrm{CS}} \qquad \tau \to \frac{1}{k}$$

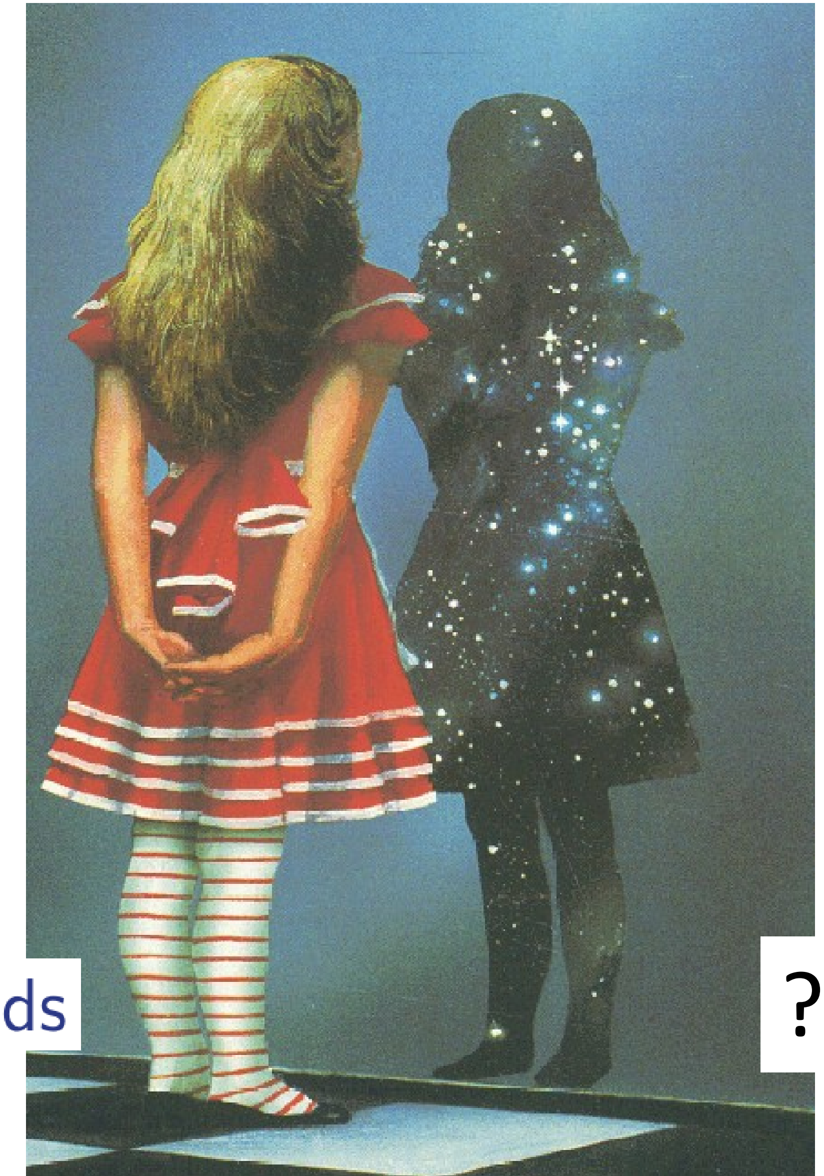
From $(k \leftrightarrow -k) \Leftrightarrow (\tau \leftrightarrow -\tau) \Leftrightarrow (q \leftrightarrow q^{-1})$, we expect

$$\boxed{\widehat{Z}_a(-M_3; \tau) = \widehat{Z}_a(M_3; -\tau)}$$

But what's this? Can we define $\widehat{Z}_a(M_3; \tau)$ *for both* $(|q| < 1 \Leftrightarrow \tau \in \mathbb{H})$ *and* $(|q| > 1 \Leftrightarrow \tau \in \mathbb{H}_-)$ *?*

Going to the Other Side





False 3-Manifolds

????????

Troubles with Thetas

$$\widehat{Z}_a(M_{3,\Gamma};\tau) := (\pm)\, q^{\Delta} \oint \prod_{v\in V} \frac{dz_v}{2\pi i z_v}\, \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)}\, \Theta^M_a(\tau;\mathbf{z})$$

$$\Theta^M_a(\tau;\mathbf{z}) := \sum_{\ell\in a+2M\mathbb{Z}^{|V|}} q^{-(\ell,M^{-1}\ell)}\mathbf{z}^{\ell}$$

$\downarrow$

$M_3 \leftrightarrow -M_3 \Leftrightarrow q \leftrightarrow q^{-1} \Leftrightarrow$ flipping the lattice signature $M \leftrightarrow -M$

no longer convergent for $|q| < 1$!

The definition for $\widehat{Z}_a(\tau)$ no longer applies after $M_3 \to -M_3$.

A Small Miracle

Recall

$$\widetilde{shad(F_0)}(\tau) = \sum_{\substack{i \in \mathbb{Z}/42 \\ i^2 \equiv 1 \ (42)}} \left(\frac{i}{21}\right) \widetilde{\theta^1_{42,i}}(\tau) = q^{-\frac{83}{168}} \hat{Z}_0(\Sigma(2,3,7),\tau)$$

It admits an expression as q-hypergeometric series

$$= q^{\frac{1}{168}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{\prod_{k=1}^{n}(1-q^{n+k})}$$

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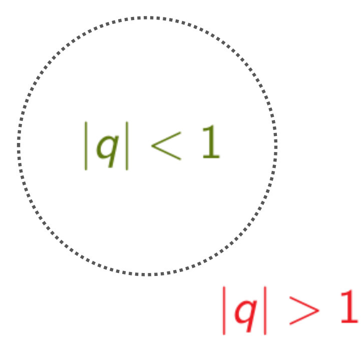
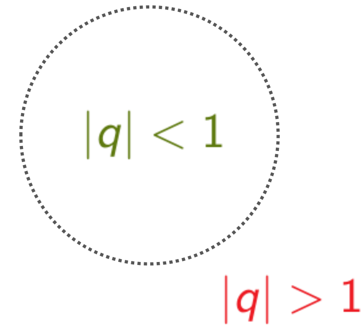
$$= q^{\frac{1}{168}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{\prod_{k=1}^{n}(1-q^{n+k})}$$

which moreover converges both inside and outside (but not on) the unit circle:

$$= q^{\frac{1}{168}} \sum_{n=0}^{\infty} \frac{q^{-n^2}}{\prod_{k=1}^{n}(1-q^{-(n+k)})}$$



A Small Miracle



$$q^{-\frac{83}{168}} \hat{Z}_0(\Sigma(2,3,7),\tau) = \sum_{\substack{i \in \mathbb{Z}/42 \\ i^2 \equiv 1 \ (42)}} \left(\frac{i}{21}\right) \widetilde{\theta^1_{42,i}}(\tau)$$

$$= q^{\frac{1}{168}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{\prod_{k=1}^{n}(1-q^{n+k})} = q^{\frac{1}{168}} \sum_{n=0}^{\infty} \frac{q^{-n^2}}{\prod_{k=1}^{n}(1-q^{-(n+k)})}$$

cf. Ramanujan's mock theta function

$$F_0(\tau) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-q^{n+k})} = 1 + q + q^3 + q^4 + O(q^5)$$

A Small Miracle

The q-hypergeometric series defines a function $F : \mathbb{H} \cup \mathbb{H}^- \to \mathbb{C}$, satisfying

$$F(\tau) = \begin{cases} \widetilde{shad(F_0)}(\tau) & \text{when } \tau \in \mathbb{H} \\ F_0(-\tau) & \text{when } \tau \in \mathbb{H}^-. \end{cases}$$

Moreover, it gives the same asymptotic expansion as $\tau \to \pm it$
$\Rightarrow$ they lead to the same *quantum modular form*.

Conjecture:

$$\hat{Z}_0(-\Sigma(2,3,7),\tau) = \hat{Z}_0(\Sigma(2,3,7),-\tau)$$

$$= q^{-\frac{1}{2}} F_0(\tau) = q^{-\frac{1}{2}}(1 + q + q^3 + q^4 + O(q^5))$$

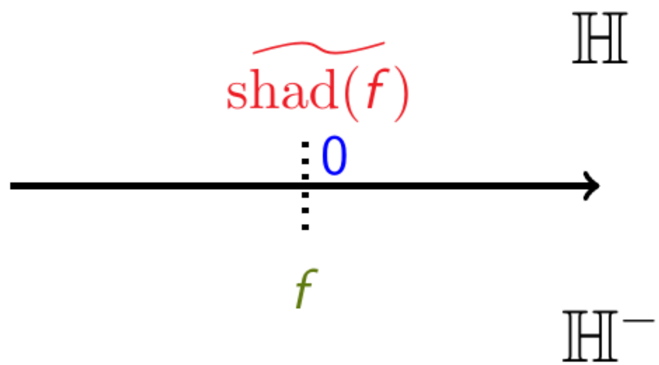
The False–Mock Conjecture

Theorem :* [MC–Duncan '13, Rhoads '18] A Rademacher sum (a regularised sum over $SL_2(\mathbb{Z})$ images) defines a function F in $\mathbb{H}$ and $\mathbb{H}^-$, satisfying

false

$$F(\tau) = \begin{cases} \widetilde{\mathrm{shad}(f)}(\tau) & \text{when } \tau \in \mathbb{H} \\ f(-\tau) & \text{when } \tau \in \mathbb{H}^-. \end{cases}$$

mock



$*$ at weight 1/2.

The False–Mock Conjecture

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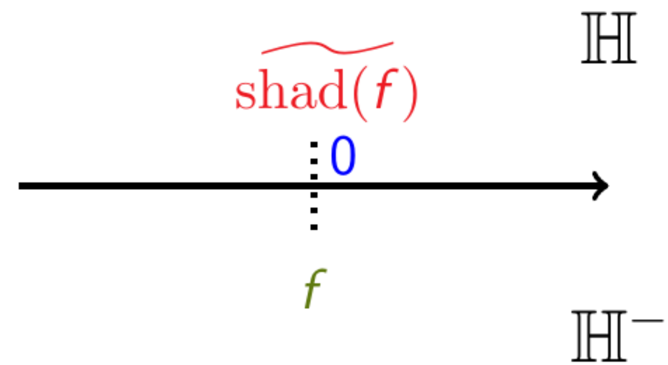


The False–Mock Conjecture: [CCFGH'18]

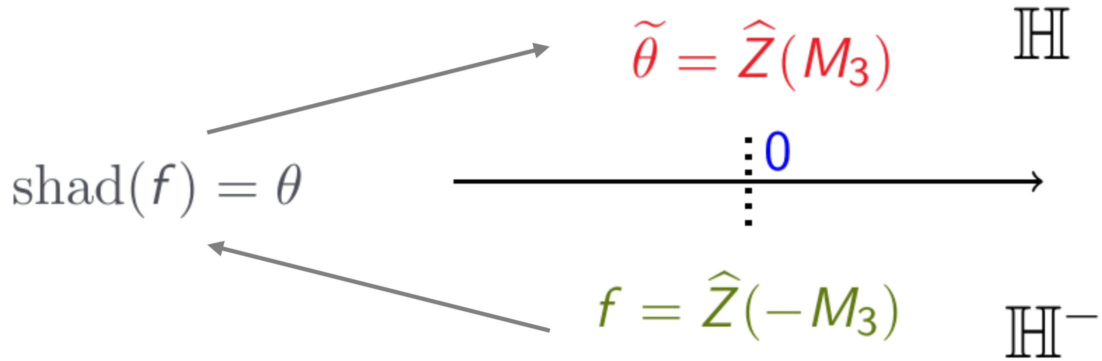
If $q^{-c}\widehat{Z}_a(M_3;\tau) = \widetilde{\theta}(\tau)$ for some $c \in \mathbb{Q}$ is a false theta function, then

$$\boxed{q^c \widehat{Z}_a(-M_3;\tau) = f(\tau)}$$

is a mock theta function with $\mathrm{shad}(f) = \theta$.

$*$ at weight 1/2.

The False–Mock Conjecture



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False–Mock Conjecture: A Test Case

Conjecture:

$$\hat{Z}_0(-\Sigma(2,3,7),\tau) = \hat{Z}_0(\Sigma(2,3,7),-\tau)$$

$$= q^{-\frac{1}{2}} F_0(\tau) = q^{-\frac{1}{2}}(1 + q + q^3 + q^4 + O(q^5))$$

Independent verification: [Gukov-Manolescu '19]

Using $-\Sigma(2,3,7) = S^3_{-1}(\text{figure } 8)$ and the surgery formula, one obtains

$$\widehat{Z}_0(-\Sigma(2,3,7),\tau) = q^{-\frac{1}{2}}(1 + q + q^3 + q^4 + q^5 + 2q^7 + \dots)$$

Defining $\widehat{Z}_a(-M_3)$

$$\widehat{Z}_a(M_{3,\Gamma};\tau) := (\pm)\, q^{\Delta} \oint \prod_{v\in V} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta^M_a(\tau;\mathbf{z})$$

$$\Theta^M_a(\tau;\mathbf{z}) := \sum_{\ell\in a+2M\mathbb{Z}^{|V|}} q^{-(\ell,M^{-1}\ell)}\mathbf{z}^{\ell}$$

$\downarrow$

$M_3 \leftrightarrow -M_3 \Leftrightarrow q \leftrightarrow q^{-1} \Leftrightarrow$ flipping the lattice signature $M \leftrightarrow -M$

no longer convergent for $|q| < 1$!

Regularised θ-function: [Zwegers '02]

$$\Theta^{-M,\mathrm{reg}}_a(\tau;\mathbf{z}) := \sum_{\ell\in a+2M\mathbb{Z}^{|V|}} \rho(\ell)\, q^{+(\ell,M^{-1}\ell)}\mathbf{z}^{\ell}$$

Defining $\widehat{Z}_a(-M_3)$

Regularised θ-function:

$$\Theta_a^{-M,\mathrm{reg}}(\tau;\mathbf{z}) := \sum_{\ell \in a + 2M\mathbb{Z}^{|V|}} \rho(\ell)\, q^{+(\ell, M^{-1}\ell)} \mathbf{z}^{\ell}$$

$$\boxed{\widehat{Z}_a(-M_{3,\Gamma};q) := (\pm)\, q^{\Delta} \oint \prod_{v \in V} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta_a^{-M,\mathrm{reg}}(\tau;\mathbf{z})}$$

[MC-Sgroi, to appear]

Using the above definition:

$$\widehat{Z}_0(-\Sigma(2,3,7),\tau) = q^{-\frac{1}{2}}(1 + q + q^3 + q^4 + q^5 + 2q^7 + \dots)$$

What we have seen:

- Explicit examples of quantum modular forms play the role of 3-manifold invariants;
- How modularity considerations lead to new examples of q-invariants consistent with physics.

III. Beyond This Talk

- More interesting modular objects beyond $SU(2)$ and beyond 3-stars:

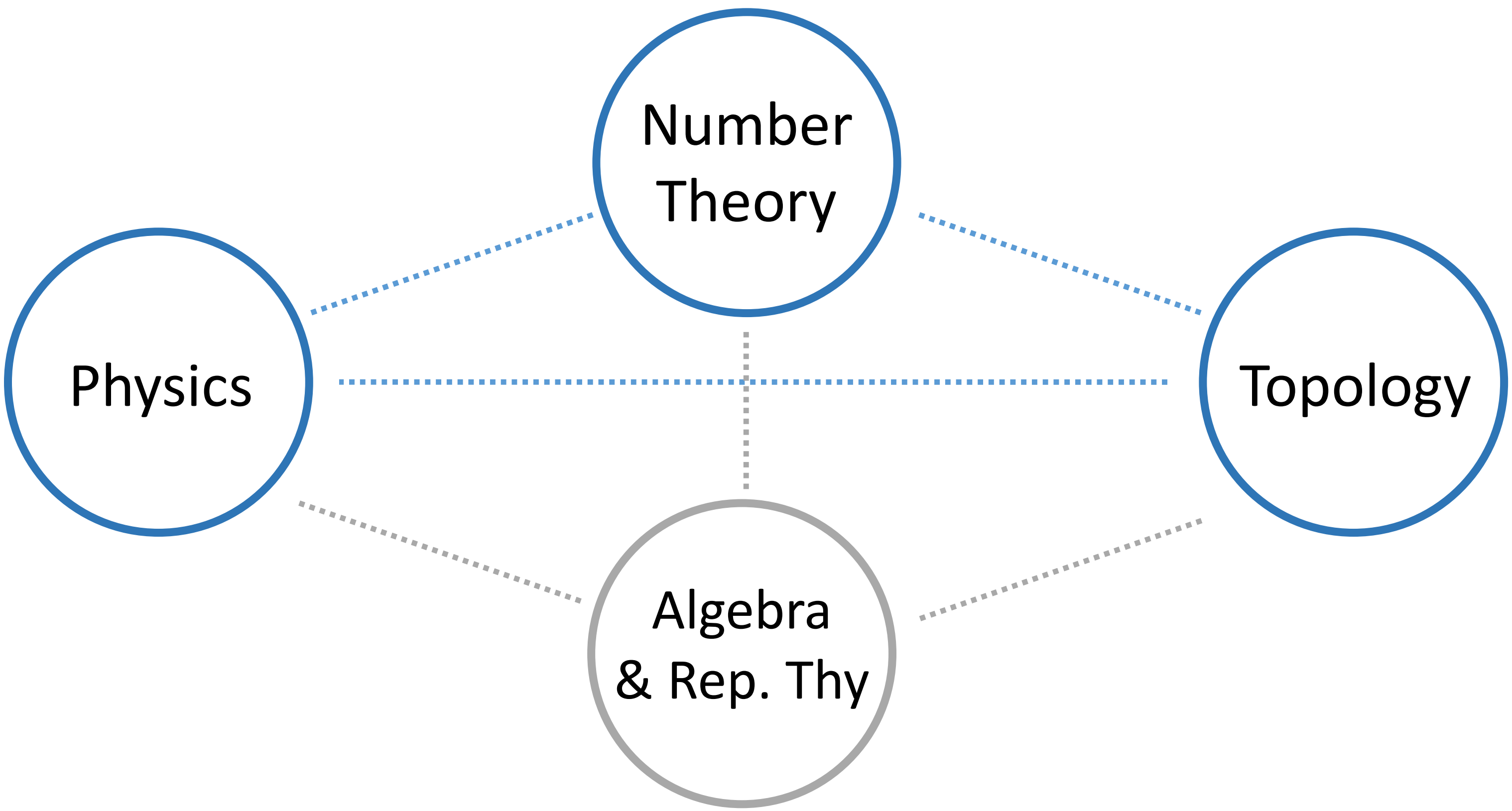
- *3d Modularity: Higher and Deeper*, to appear
with S. Chun, B. Feigin, F. Ferrari, S. Gukov, S. Harrison

- [Bringmann-Mahlburg-Milas '19]

- ...

• We also observed a close relation between $\widehat{Z}_a(M_3;\tau)$ and characters of **logarithmic vertex operator algebras**.

$$(M_3, G) \leftrightarrow \text{log VOA } \mathcal{V}^G(M_3)$$



Tack så mycket!