Realizing the potential of generalized Kähler geometry

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and Maxim Zabzine

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Morita equivalence and the generalized Kähler potential

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Abstract

We solve the problem of determining the fundamental degrees of freedom underlying a generalized Kähler structure of symplectic type. For a usual Kähler structure, it is well-known that the geometry is determined by a complex structure, a Kähler class, and the choice of a positive $(1,1)$-form in this class, which depends locally on only a single real-valued function: the Kähler potential. Such a description for generalized Kähler geometry has been sought since it was discovered in 1984. We show that a generalized Kähler structure of symplectic type is determined by a pair of holomorphic Poisson manifolds, a holomorphic symplectic Morita equivalence between them, and the choice of a positive Lagrangian brane bisection, which depends locally on only a single real-valued function, which we call the generalized Kähler potential. Our solution draws upon, and specializes to, the many results in the physics literature which solve the problem under the assumption (which we do not make) that the Poisson structures involved have constant rank. To solve the problem we make use of, and generalize, two main tools: the first is the notion of symplectic Morita equivalence, developed by Weinstein and Xu to study Poisson manifolds; the second is Donaldson's interpretation of a Kähler metric as a real Lagrangian submanifold in a deformation of the holomorphic cotangent bundle.

1979 Zumino: The 2d σ-model

$$\left\{ (\Sigma^2, h) \xrightarrow{\varphi} (M^n, g) \right\}$$ h, g metrics

Inherits $\mathcal{N}=(2,2)$ susy from Kähler $I : TM \circlearrowleft$ $I^2 = -1$

1984 Gates-Hull-Roček: same is true for generalized Kähler

DEF (I_+, I_-) g-compatible complex structures

s.t. $\boxed{d^c_+ \omega_+ + d^c_- \omega_- = 0}$, $\boxed{d d^c_\pm \omega_\pm = 0}$

ω_+, ω_- Hermitian forms

Kähler potential

$$\boxed{g_{i\bar{j}} = \frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j} K}$$

$(U, (z_1, \dots, z_n))$ complex chart, $K \in C^\infty(U, \mathbb{R})$

Zumino: extend $\varphi : \Sigma \to M$
to superfield $\Phi : \tilde{\Sigma} \to M$

Then action is $\int_{\tilde{\Sigma}} \Phi^* K \, d\mathrm{vol}_{\tilde{\Sigma}}$

Q1: analog of K for Gen. Kähler ?

Quantization

for I kähler, $\omega = gI$ may Pre-Quantize

to $(L, \|\cdot\|, \nabla)$

line bundle

Hermitian

unitary with $F(\nabla) = i\omega$

$\Rightarrow \mathcal{H} = H^0(M, L)$ using $\bar{\partial}_L = \nabla^{0,1}$

$$\boxed{\mathcal{A} = H^0(M, \underline{\mathbb{C}} \oplus L \oplus L^2 \oplus \cdots)}$$ $\mathbb{Z}$-graded algebra

$\boxed{\text{Q2: Analog of } \mathcal{H} \text{ and } \mathcal{A} \text{ for Gen Kähler?}}$

Main tool: GENERALIZED COMPLEX structure (Hitchin)

$$\mathbb{I} : T \oplus T^* \longrightarrow T \oplus T^* \qquad \mathbb{I}^2 = -1$$

Involutive for the Courant bracket twisted by 3-form H, $dH=0$.

- ω symplectic $\Rightarrow$ $\mathbb{I} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$
- I complex $\Rightarrow$ $\mathbb{I} = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}$
- I complex
  $\sigma = IQ + iQ$
  holom. Poisson
  $\Rightarrow$ $\mathbb{I} = \begin{pmatrix} I & Q \\ 0 & -I^* \end{pmatrix}$

Theorem (MG) Given (g, I_+, I_-) such that $d_+^c\omega_+ + d_-^c\omega_- = 0$ and $H = d_+^c\omega_+$ is closed $(dd_\pm^c\omega_\pm = 0)$,

$$\mathbb{I}_\pm = \frac{1}{2}\begin{pmatrix} I_+ \pm I_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(I_+^* \pm I_-^*) \end{pmatrix}$$

defines a pair of commuting generalized complex structures with

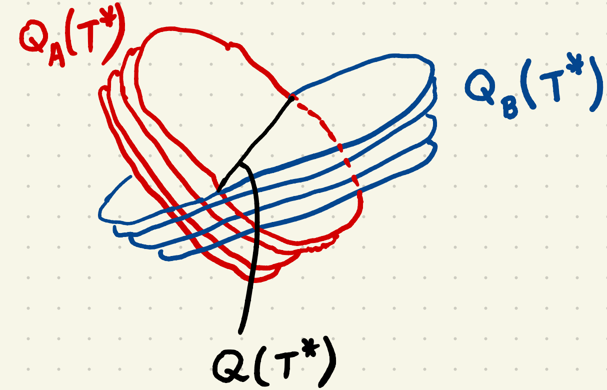
$$\mathbb{G}(-,-) = \langle \mathbb{I}_+ -, \mathbb{I}_- - \rangle$$

positive definite.

Furthermore, this is an EQUIVALENCE

GK and Poisson geometry:

$$\left.\begin{aligned}(I_+ + I_-)\,g^{-1} &= Q_A\\ (I_+ - I_-)\,g^{-1} &= Q_B\\ [I_+, I_-]\,g^{-1} &= Q\end{aligned}\right\}\ \text{Poisson structures}$$



$$\boxed{\sigma_\pm = I_\pm Q + iQ \quad \text{holom. Poisson (Hitchin)}}$$

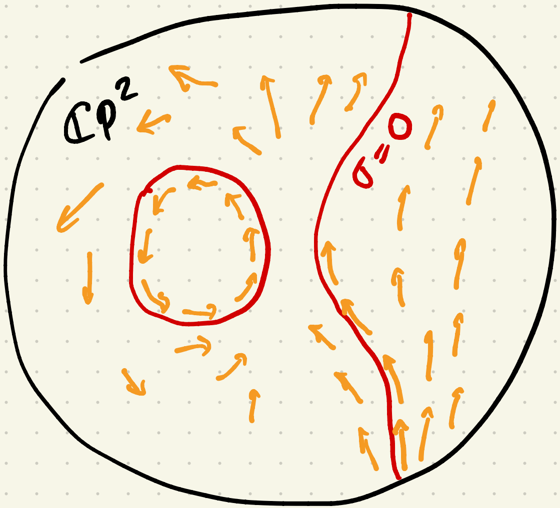
In the usual Kähler case, $Q_A = \omega^{-1}$ and $\sigma_\pm = 0$

$Q_B = 0$

Deform a Kähler structure to a GK with $\sigma_\pm \neq 0$

Construction of GK :

$$\begin{cases} (M, I_-) = \mathbb{C}P^2 \\ \sigma \in H^0(\Lambda^2 T) = H^0(\mathbb{P}^2, \mathcal{O}(3)) \\ \omega_0 = \text{Fubini-Study Kähler} \end{cases}$$



$$\left.\begin{matrix} \sigma \in H^0(\Lambda^2 T) \\ \omega_0 \in H^1(\Omega^1) \end{matrix}\right\} \quad [\sigma(\omega_0)] \in H^1(T) = 0$$

$$\Rightarrow \exists \; v \in C^\infty(TM) \text{ s.t. } \begin{cases} \bar\partial v = \sigma(\omega_0) \\ [v, \sigma] = 0 \end{cases}$$

$$I_+ = \varphi_1^v(I_-) \qquad g = -\bar\omega\left(\frac{I_+ + I_-}{2}\right) \qquad \bar\omega = \int_0^1 (\varphi_s^v)^* \omega_0 \, ds$$

$$\boxed{(g, I_+, I_-) \quad \text{Generalized Kähler on } \mathbb{C}P^2, \; \sigma_- = \sigma}$$

Summary: want Gen. Kähler potential K

want a (Pre) - Quantization $\mathcal{H}, \mathcal{A}$

key geometrical input: holomorphic Poisson geometry $\sigma_\pm$

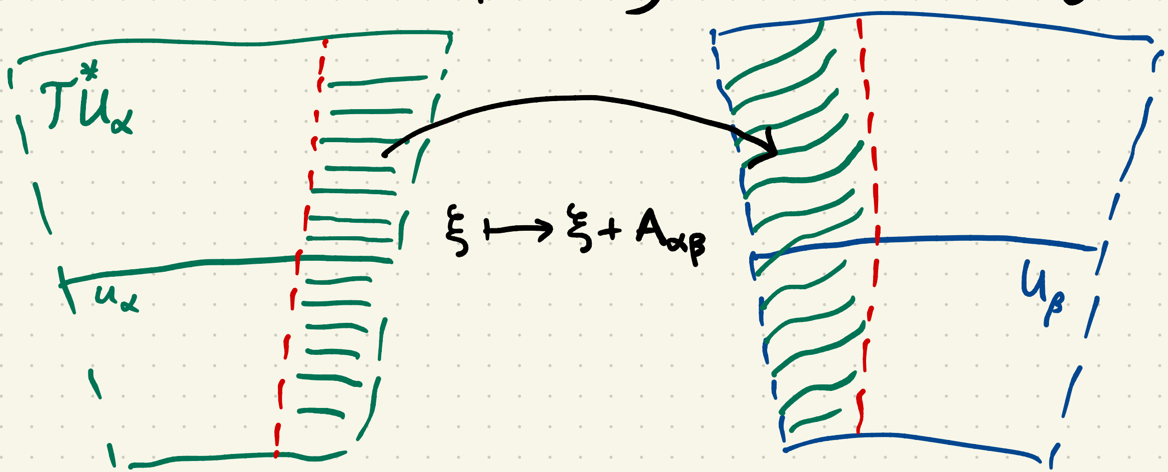
The Kähler potential according to (Donaldson '01)

$$\omega = i\partial\bar\partial K_\alpha = \bar\partial(\underbrace{-i\partial K_\alpha}_{A_\alpha}) = \bar\partial A_\alpha \,, \quad A_\alpha \in \Omega^{1,0}(U_\alpha)$$

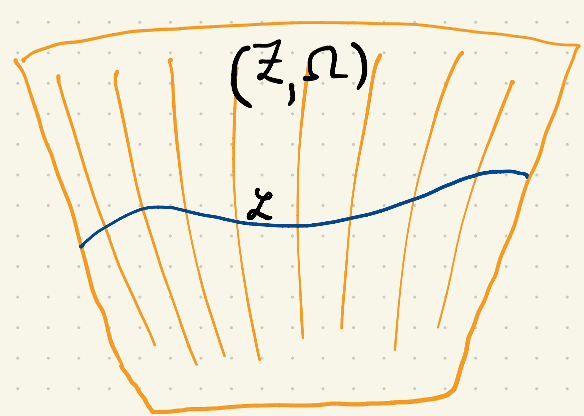
$A_{\alpha\beta} = A_\alpha - A_\beta$ holomorphic

$$\boxed{[A_{\alpha\beta}] \in H^1(\Omega^1) \quad \text{Kähler class}}$$

Glue T^*U_α to T^*U_β using translation by $A_{\alpha\beta}$



Result $\quad Z = \coprod_\alpha T^*\mathcal{U}_\alpha \big/ (\xi \sim \xi + A_{\alpha\beta})$



- affine bundle for T^*X
- holomorphic symplectic form Ω
- C^∞ section $\mathcal{L} = \text{graph}(-i\partial K_\alpha)$

with $\Omega|_{\mathcal{L}} = d(-i\partial K_\alpha) = \omega$ Real.

$\mathcal{L}$ is $\begin{cases} \text{Lagrangian for } \operatorname{Im}\Omega \\ \text{Symplectic for } \operatorname{Re}\Omega \end{cases}$

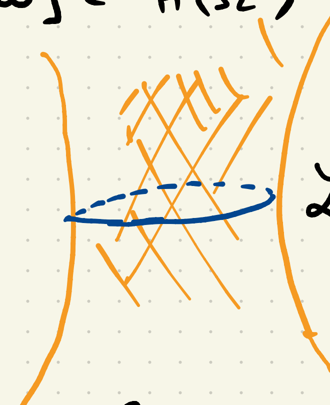
Global meaning for Kähler potential: $\boxed{\mathcal{L} \text{ is an A-brane for } \operatorname{Im}\Omega}$

Ex.: $(\mathbb{P}^1, \omega_{FS})$

Z = affine bundle with class $[\omega] \in H^1(\Omega^1)$

$\cong$ affine quadric surface

$$\{x^2+y^2+z^2=1\} \subset \mathbb{C}^3$$



$\mathcal{L}$

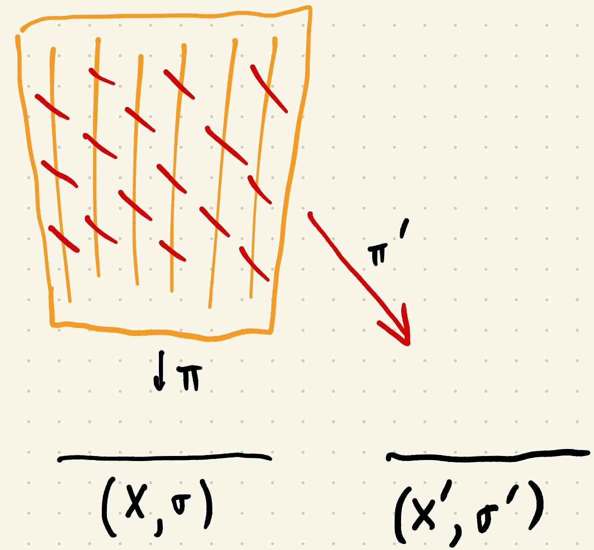
$$\Omega = \frac{dx \wedge dy}{2z}$$

$\mathcal{L}$ = Real locus $\{(x,y,z) = (\bar{x},\bar{y},\bar{z})\}$

Morita category

Weinstein: study poisson (X,σ) via its symplectic realizations

$$\begin{array}{c} (Z,\Omega) \\ \downarrow{\scriptstyle \pi} \\ (X,\sigma) \end{array} \qquad \pi_* \Omega^{-1} = \sigma$$



- $(\ker \pi_*)^{\perp_\Omega}$ Involutive
- Quotient is Poisson (X', σ')

Morita 2-category (Weinstein)

$$\mathrm{Mor}^1(X, X') = \begin{tikzcd}[column sep=small] & (Z,\Omega) \arrow[dl, "\pi"', "\text{symp.}"] \arrow[dr, "\pi'", "\text{anti-symp}"'] & \\ (X,\sigma) & & (X',\sigma') \end{tikzcd}$$

s.t. $\ker \pi_* \perp^{\Omega} \ker \pi'_*$ i.e. $\{\mathcal{O}_X, \mathcal{O}_{X'}\} = 0$

$$\mathrm{Mor}^2(Z, Z') = \text{isos } (Z,\Omega) \to (Z',\Omega')$$

$$\mathrm{Mor\,Pic}(X,\sigma) = \left\{ \begin{tikzcd} (Z,\Omega) \arrow[d, shift right] \arrow[d, shift left] \\ (X,\sigma) \end{tikzcd} \right\} \Big/ \cong$$

Picard group.

Identity in Pic is a distinguished self-equivalence: Weinstein groupoid.

Ex.: $\mathfrak{g}$ Lie algebra $\Rightarrow$ $(\mathfrak{g}^*, \sigma_{KKS})$

G Lie group integrating $\mathfrak{g}$

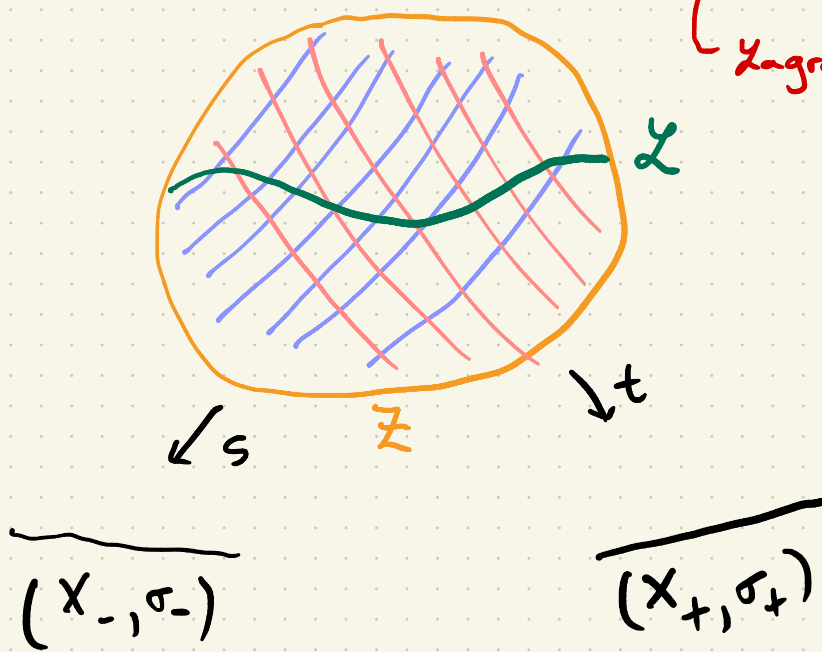
$$Z = T^*G \ , \quad \Omega = \Omega_{can}$$

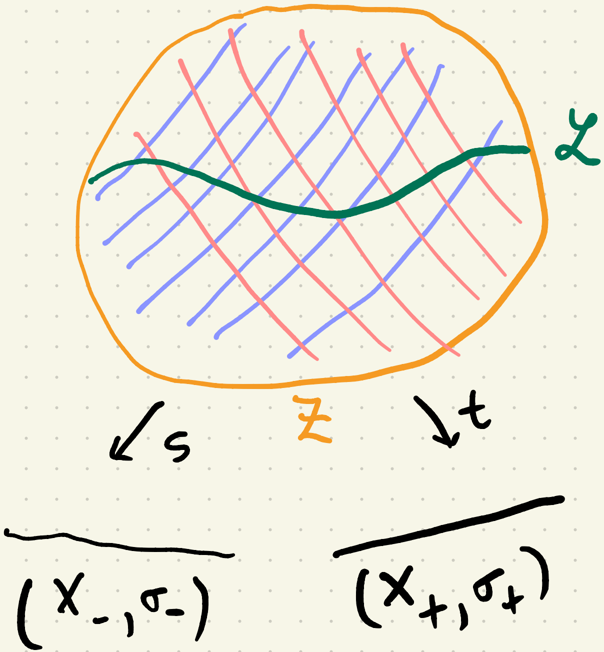
$$\boxed{\begin{tikzcd} (T^*G, \Omega_{can}) \arrow[d, shift right, "L^*"'] \arrow[d, shift left, "R^*"] \\ (\mathfrak{g}^*, \sigma_{KKS}) \end{tikzcd}}$$

Theorem (Bischoff, M.G., Zabzine) (Assuming $Q_A^{-1} = F$ exists)

A Generalized Kähler structure (g, I_+, I_-) is equivalent to a Hol. symplectic Morita equivalence $(Z, \Omega) : (X_-, \sigma_-) \to (X_+, \sigma_+)$ together with a nondegenerate C^∞ brane bisection $\mathcal{L} \subset Z$.

- section of s + t
- Lagrangian for $\operatorname{Im}\Omega$





- Diffeo $X_- \cong \mathcal{L} \cong X_+$
- s, t foliations induce I_+, I_- on $\mathcal{L}$.
- $\Omega|_{\mathcal{L}} = F$ real and

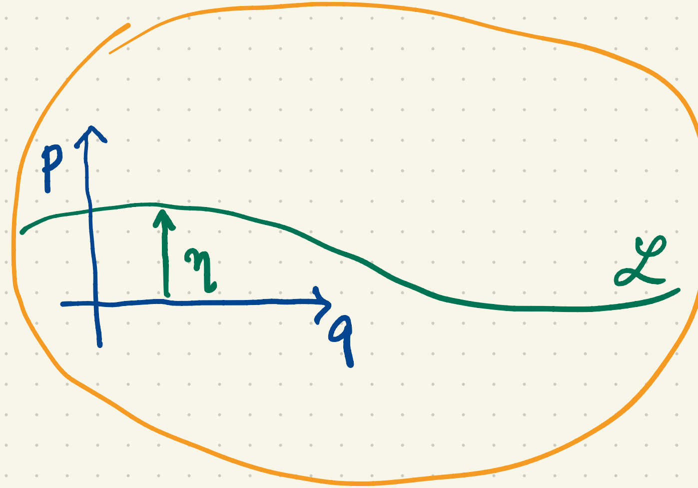
$$F^{(1,1)_+} I_+ = F^{(1,1)_-} I_- = g$$

unique symmetric tensor S^2T^*

require Riemannian.

Potential function

In holomorphic Darboux chart



$$\mathcal{L} = \mathrm{graph}(\eta) \qquad \eta^{1,0}$$

$$d\eta = \Omega|_{\mathcal{L}} \quad \text{is } \underline{\text{real}}$$

$$\mathrm{Im}(\eta) = dK \qquad K \in C^\infty(U, \mathbb{R})$$

$$\Rightarrow \eta = 2i\partial K \Rightarrow \boxed{d\eta = \Omega|_{\mathcal{L}} = -2i\partial\bar{\partial}K}$$

$\Rightarrow$ g determined by real smooth function K.

Ex.: $X = \mathbb{C}^2$, $\sigma = t\, x\partial_x \wedge y\partial_y$

$Z = \mathbb{C}^2 \times \mathbb{C}^2 \ni (x, y, a, b)$

$s = (e^{-tby} x, y)$ $\qquad t = (x, e^{-tax} y)$

$\Omega = (t^* - s^*) t^{-1} \frac{dx}{x} \wedge \frac{dy}{y} = da \wedge dx + db \wedge dy$

choose $K = K(x, \bar{x}, y, \bar{y})$, can choose T^2-invariant

e.g.

$K = K(|x|^2, |y|^2) = \frac{n}{2\pi} \log(1 + |x|^2) + \frac{m}{2\pi} \log(1 + |y|^2)$

(for $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{O}(n, m)$)

$= \frac{1}{2\pi} \log(1 + |x|^2 + |y|^2)$

(for $\mathbb{CP}^2$, $\mathcal{O}(1)$)

these will define toric GK str. for t small

Quantization

Branes And Quantization

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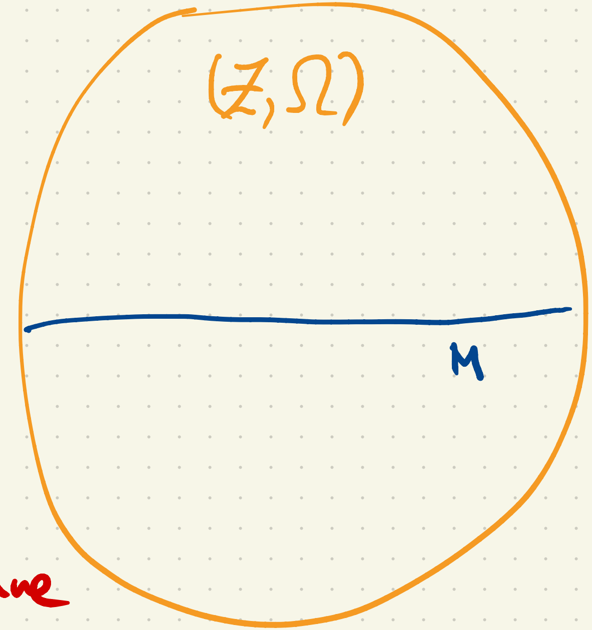
The problem of quantizing a symplectic manifold (M, ω) can be formulated in terms of the A-model of a complexification of M. This leads to an interesting new perspective on quantization. From this point of view, the Hilbert space obtained by quantization of (M, ω) is the space of $(\mathcal{B}_{cc}, \mathcal{B}')$ strings, where $\mathcal{B}_{cc}$ and $\mathcal{B}'$ are two A-branes; $\mathcal{B}'$ is an ordinary Lagrangian A-brane, and $\mathcal{B}_{cc}$ is a space-filling coisotropic A-brane. $\mathcal{B}'$ is supported on M, and the choice of ω is encoded in the choice of $\mathcal{B}_{cc}$. As an example, we describe from this point of view the representations of the group $SL(2, \mathbb{R})$. Another application is to Chern-Simons gauge theory.

Gukov-Witten: embed (M,ω) into (Z,Ω)

s.t. $\Omega\big|_M = \omega$

i.e. M is
- Lagrangian for $\mathrm{Im}\,\Omega$
- symplectic for $\mathrm{Re}\,\Omega$



$\Rightarrow$ two branes M, Z_{cc}

space filling brane

Lagrangian brane

$$\Rightarrow \boxed{\mathcal{H} = \mathrm{Hom}(M, Z_{cc})}$$

Proposal for quantization

$$\mathcal{L} \subset (\mathcal{Z}, \Omega)$$

brane bisection

Morita equivalence.

: two branes in A-model of $(\mathcal{Z}, \operatorname{Im}\Omega)$

$$\boxed{\mathcal{H} = \operatorname{Hom}(\mathcal{L}, \mathcal{Z}_{cc})}$$

How do we get an <u>algebra</u> from this?

Composition of Morita maps:

$$\begin{tikzcd}
 & Z_{01} \arrow[dl] \arrow[dr] & & Z_{12} \arrow[dl] \arrow[dr] & \\
X_0 & & X_1 & & X_2 \\
 & & Z_{02} \arrow[ull] \arrow[urr] & &
\end{tikzcd}$$

coisotropic $\qquad\qquad \Omega_{01} \times \Omega_{12}$

$$\begin{tikzcd}
Z_{01} \times_{X_1} Z_{12} \arrow[r, hook] \arrow[d, "\text{reduce}"'] & Z_{01} \times Z_{12} \\
Z_{02} &
\end{tikzcd}$$

$\Rightarrow$ Lagrangian relation $\qquad Z_{01} \times_{X_1} Z_{12} \longrightarrow \overline{Z_{01} \times Z_{12}} \times Z_{02}$

$\Rightarrow$ morphism $\qquad \mathrm{Fuk}(Z_{01}) \times \mathrm{Fuk}(Z_{12}) \xrightarrow[\Phi]{} \mathrm{Fuk}(Z_{02})$

If $\mathcal{Z} \in \mathrm{MorPic}(X)$, then we take powers

$(\mathcal{G}, \mathcal{Z}, \mathcal{Z}^2, \mathcal{Z}^3, \ldots)$ sequence of hol. symplectic mfd

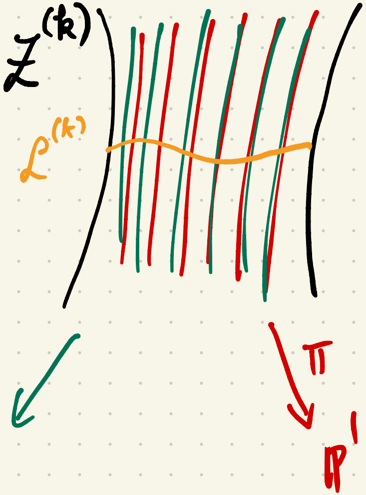
$\mathcal{Z}^k$ has two branes $\mathcal{L}^k, \mathcal{Z}^k_{cc}$

$$\Rightarrow \boxed{A^k = \mathrm{Hom}_{\mathcal{Z}^k}(\mathcal{L}^k, \mathcal{Z}^k_{cc})}$$

composition $\Rightarrow$ $\Phi : (\mathcal{L}^k, \mathcal{L}^l) \longmapsto \mathcal{L}^{k+l}$

$$\boxed{\Phi_* : \mathrm{Hom}(\mathcal{L}^k, \mathcal{Z}^k_{cc}) \otimes \mathrm{Hom}(\mathcal{L}^l, \mathcal{Z}^l_{cc}) \longrightarrow \mathrm{Hom}(\mathcal{L}^{k+l}, \mathcal{Z}^{k+l}_{cc})}$$

Ex.:



choose $\varphi \in \mathrm{Aut}\,\mathbb{P}^1$ and $\mathcal{O}(1) \to \mathbb{P}^1$.

$Z^{(k)} =$ k-twisted cotangent $\Omega = \Omega_{can} + \pi^* k\,\omega_{FS}$

$s = \pi$

$t = \varphi^k \circ \pi$

$L^{(k)} =$ C^∞ zero section.

- Prequantize Ω to $D = \pi^* \nabla^{\mathcal{O}(k)} + i\Theta$ hol. connection with $F(D) = \Omega$.

Compute $\mathrm{Hom}_{\mathcal{F}(\mathrm{Im}\,\Omega)}(L^{(k)}, Z^{(k)}_{\alpha})$ via hol. Lag. Polariz. of Z

obtain a Hol. sheaf on L. $\Rightarrow$ $H^0(\mathbb{P}^1, \mathcal{O}(k))$

Product: $$\boxed{f^{(k)} * g^{(\ell)} = f \otimes \varphi^k(g)}$$ Van den Bergh twisted coord. ring.

eg.

$$\left.\begin{aligned} x * y &= e^{h/2}\, x y \\ y * x &= e^{-h/2}\, y x \end{aligned}\right\} \quad x * y = e^{h}\, y * x$$

generalize to Toric Poisson varieties X

- Explicit Morita self-equivalence: Z = twisted T^*X

- Iteration $\{Z^{(k)}\}_{k\in\mathbb{Z}}$ each with $Z^{(k)}_{cc}$, $\mathcal{L}^{(k)}$

- $\mathrm{Hom}_{Z^{(k)}}(\mathcal{L}^{(k)}, Z^{(k)}_{cc})$ is a hol. sheaf on X

- obtain a sheaf of NC algebras deforming $\mathcal{O}_X$;
  but topology = torus-invariant opens.