Chiral central charge and Thermal Hall effect on a lattice

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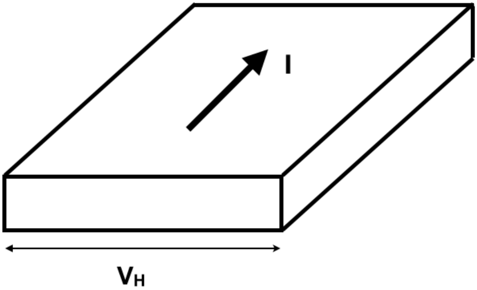
Outline

- Bulk-boundary correspondence for the Quantum Hall Effect
- Thermal Hall Effect vs QHE effect: similarities and differences
- Chiral anomalies and equilibrium currents
- Kubo formulas for Hall conductance and Thermal Hall conductance
- Thermal Hall Conductance as a relative transport coefficient

Based on two papers with Lev Spodyneiko (Caltech): 1904.05491 and 1905.06488

Hall Effect



$$j_x = \sigma_{xy}(T) E_y$$

$$I = \sigma_{xy}(T) V_H$$

Quantum Hall Effect

If there is an energy gap Δ between the ground state and excited states, then for $T \ll \Delta$ the Hall conductance is "quantized":

$$\sigma_{xy}(0) \simeq \frac{k}{2\pi N}, \quad k \in \mathbb{Z}$$

where N is the number of ground states on a torus ($N = 1$ for IQHE).

At $T = 0$ there is no dissipation, and the Hall effect can be described by a classical field theory (Chern-Simons theory):

$$S = \frac{1}{2}\sigma_{xy}(0) \int AdA.$$

Robustness of $\sigma_{xy}(0)$ (invariance under deformations of the Hamiltonian which do not close the gap) can be explained by non-renormalization of the Chern-Simons action.

Boundary-bulk correspondence for QHE

When $\sigma_{xy}(0) \neq 0$, there have to be gapless edge modes. If we assume they are described by a 1+1d unitary CFT, then there must be a $U(1)$ current algebra such that

$$\frac{k_R - k_L}{2\pi} = \sigma_{xy}(0).$$

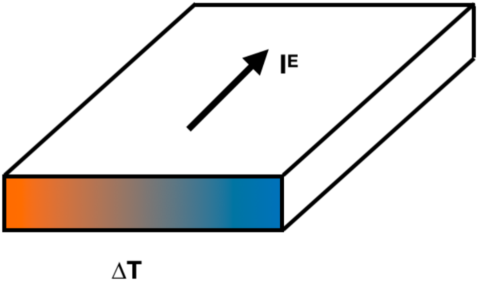
Here k_R (resp. k_L) is the level of the holomorphic (resp. anti-holomorphic) $U(1)$ current algebra:

$$j(z)j(w) \sim \frac{k_R/2\pi}{(z-w)^2}.$$

$k_R - k_L$ measures 't Hooft anomaly for $U(1)$ symmetry. This provides another explanation for the robustness of $\sigma_{xy}(0)$.

The anomaly inflow argument implies that $k_R - k_L$ is independent of the choice of the edge.

Thermal Hall Effect



$$\langle j_x^E \rangle = \kappa_{xy}(T)\partial_y T$$

$$\langle I^E \rangle = \kappa_{xy}(T)\Delta T$$

Thermal Hall Lore

- In a gapped 2d system one must have $\kappa_{xy}(T) = aT + O(e^{-\Delta/T})$.
- The coefficient a is robust under deformations of the Hamiltonian which do not close the energy gap.
- The coefficient a is related to the chiral central charge of the edge modes:

$$a = \frac{\pi}{6}(c_R - c_L).$$

- Thermal Hall Effect can be described by a gravitational Chern-Simons action

$$S_{grav} \sim \int \mathrm{Tr}\left(\omega d\omega + \frac{2}{3}\omega^3\right).$$

Bulk-boundary correspondence for $c_R - c_L$

Consider a strip $0 \leq y \leq L$ with edges at temperatures T_0 and T_1. Assume $|T_1 - T_0| \ll \frac{1}{2}(T_1 + T_0)$. Assume all temperatures are much smaller than the bulk energy gap Δ.

The energy current arises from the edge modes. Modular anomaly implies that the equilibrium energy current in a CFT is

$$\langle I^E \rangle_{CFT} = \frac{\pi}{12}(c_R - c_L) T^2.$$

The net energy current in the x direction is

$$\langle I_x^E \rangle \simeq \frac{\pi}{12}(c_R - c_L)(T_1^2 - T_0^2) = \frac{\pi}{6}(c_R - c_L)\frac{T_1 + T_0}{2}\Delta T.$$

Hence $\kappa_{xy}(T) \simeq \frac{\pi}{6}(c_R - c_L) T$.

Difficulties with the Thermal Hall Lore

- No natural way to couple typical cond-mat systems to a metric, so gravitational response is ill-defined.
- Chern-Simons action does not lead to Thermal Hall effect anyway (gives energy flux proportional to 2nd derivatives of T)
- Not clear why a should be robust under deformations
- Not clear why $c_R - c_L$ must be independent of the edge (the anomaly inflow argument does not work because the bulk cannot be coupled to gravity)
- No good microscopic formula for $\kappa_{xy}(T)$ which would help one to resolve these issues (there is a Kubo formula for $\sigma_{xy}(T)$).

Some answers

- Independence of $c_R - c_L$ of the choice of the edge can be shown if one assumes that a well-defined microscopic system in equilibrium cannot have a nonzero energy current
- In the case of electric currents, this is known as Bloch's theorem; the energy version of Bloch's theorem requires an entirely new argument
- Thermal Hall conductance is not a well-defined bulk transport coefficient, hence the difficulties with writing a good microscopic formula for it
- Derivatives of κ_{xy} w. r. to parameters are well-defined and one can write a formula for them
- κ_{xy}/T is not a function on the parameter space but an exact 1-form

Bloch's theorem

F. Bloch showed (1933) that in a quasi-1d system of particles one has

$$\langle I_x \rangle = 0$$

This is a pre-requisite for Ohm's law and is more general than Ohm's law (holds in superconductors and integrable systems, even though Ohm's law does not apply there).

- Can also be proved for lattice systems (H. Watanabe, 2019).
- Proof relies on charge quantization (D. Bohm, 1949)
- Does not apply to 1+1d field theories with a $U(1)$ anomaly.

Specifically, if one perturbs a 1+1d CFT with a $U(1)$ symmetry by a chemical potential μ, then

$$\langle I_x \rangle = \frac{(k_R - k_L)\mu}{2\pi}.$$

Bloch's theorem and QHE

Bloch's theorem implies that $k_R - k_L$ does not depend on the choice of the edge.

Consider a "nice" 2d system (lattice system or particles with short-range interactions) with an energy gap Δ for $T \ll \Delta$ on a strip $0 \leq y \leq L$. The edges at $y = 0$ and $y = L$ may be different.

By Bloch's theorem, $\langle I_x \rangle = 0$. On the other hand, since there are no bulk excitations, the energy current comes entirely from the two edges.

Suppose each edge is described by a CFT. Perturbing by a chemical potential must still give $\langle I_x \rangle = 0$, hence $k_R - k_L$ must be the same for the two edges.

Bloch's theorem for the energy current

It is intuitively clear that Bloch's result should also apply to the energy current. But Bloch's proof does not generalize to the energy current.

Nevertheless, one can prove it both for lattice systems and particle systems if one assumes that there are no phase transitions in 1d for $T > 0$ (Lev Spodyneiko and AK, 2019).

We use a deformation argument: show that the net energy current does not change under a large class of deformations of the Hamiltonian. Then deform the Hamiltonian to zero (for lattice systems) or to a free Hamiltonian (for particle systems).

Energy Bloch theorem and the chiral central charge

The energy Bloch theorem implies:

- A lattice system or a particle system cannot flow to a 1+1d CFT with a nonzero $c_R - c_L$
- For a 2d system with a bulk energy gap, $c_R - c_L$ for the edge modes is independent of the choice of the edge.

Can use this to give a boundary definition of a in $\kappa_{xy}(T) = aT + \ldots$:

$$a = \frac{\pi}{6}(c_R - c_L).$$

Is Thermal Hall conductance well-defined?

Thermal Hall conductance is really the skew-symmetric part of the heat conductance tensor:

$$j_i^E = -\kappa_{ij}^S \partial_j T + \epsilon_{ij}\kappa_{xy}\partial_j T.$$

It gives no contribution to the conservation equation $\frac{\partial \rho^E}{\partial t} = -\partial_k j_k^E$, since

$$\partial_i\left(\epsilon_{ij}\kappa_{xy}(T)\partial_j T\right) = 0.$$

Moreover, the contribution of κ_{xy} to the net energy current across a section of a system can be written as a boundary term:

$$\int_0^L dy\ \kappa_{xy}(T)\partial_y T = \nu(T(L)) - \nu(T(0)),$$

where $\nu(T) = \int^T \kappa_{xy}(u)du$.

Kubo formula for Hall conductance

Let $\langle ... \rangle$ denote Gibbs average at temperature $T = 1/\beta$. The Kubo pairing of two operators is defined as

$$\langle\langle A; B\rangle\rangle = \frac{1}{\beta}\int_0^\beta d\tau\ \langle e^{H\tau} A e^{-H\tau} B\rangle - \langle A\rangle\langle B\rangle.$$

Hall conductance can be computed as

$$\sigma_{xy}(T) = \frac{1}{2}\lim_{s\to 0}\lim_{V\to\infty}\frac{\beta}{V}\int_0^\infty e^{-st}\langle\langle J_x(t); J_y(0)\rangle\rangle dt - (x \leftrightarrow y).$$

Here $J_k = \int_V d^2\mathbf{r}\ j_k(\mathbf{r})$, and $A(t) = e^{iHt} A(0) e^{-iHt}$.

Kubo formula for thermal Hall conductance

The naive analog of this for thermal Hall conductance is

$$\kappa_{xy}^{Kubo}(T) = \frac{1}{2} \lim_{s\to 0} \lim_{V\to\infty} \frac{\beta^2}{V} \int_0^\infty e^{-st} \langle\langle J_x^E(t); J_y^E(0)\rangle\rangle dt - (x \leftrightarrow y).$$

But it is not correct: typically diverges as $T \to 0$ instead of vanishing.

Reason: one cannot formulate the computation on a torus, so boundary contributions are important.

A way to correct the naive Kubo formula for κ_{xy} was proposed by Cooper, Halperin and Ruzin (1997).

Energy magnetization

In equilibrium one has $\langle \partial_k j_k^E \rangle = 0$. Hence locally

$$\langle j_k^E(\mathbf{r}) \rangle = \epsilon_{kj} \partial_j M^E(\mathbf{r}).$$

$M^E(\mathbf{r})$ is defined up to a constant and is called energy magnetization.

Cooper et al. proposed a modified Kubo formula for κ_{xy}:

$$\kappa_{xy} = \kappa_{xy}^{Kubo} - 2 \lim_{V \to \infty} \frac{\beta}{V} \int M^E(\mathbf{r}) d^2\mathbf{r}. \qquad (1)$$

There are several derivations in the literature by now, usually in the context of hydrodynamics.

Problems with energy magnetization

- What if the space is not simply-connected? Can $M^E(\mathbf{r})$ become multi-valued? Equivalently, if we regard j^E as a 1-form, then $d\langle j^E\rangle = 0$. Can solve $\langle j^E\rangle = dM^E$ only if the cohomology class of $\langle j^E\rangle$ is trivial.
- If M^E is univalued, it is only defined up to a constant. How do we fix the constant?

The 1st problem actually does not arise thanks to the energy Bloch theorem.

The 2nd problem is more serious than it seems because the "constant" may depend on T and other parameters.

Energy magnetization as a 1-form

Kitaev (2006):

For lattice systems, derivatives of $M^E(\mathbf{r})$ with respect to parameters of the Hamiltonian are well-defined.

Let $\mu^E = DM^E$, where D denotes exterior derivative on the parameter space. Then

$$D\kappa_{xy} = D\kappa_{xy}^{Kubo} - 2 \lim_{V\to\infty} \frac{\beta}{V} \int \mu^E d^2\mathbf{r}$$

is a well-defined closed 1-form on the parameter space.

One can include T among the parameters using a homogeneity relation: rescaling $H \mapsto \lambda H$ and $T \mapsto \lambda T$ does not affect any expectation values.

Thermal Hall conductance as a 1-form

More precisely, since κ_{xy}^{Kubo} and $\beta\mu^E$ are homogeneous of degree 1 under $H \mapsto \lambda H$, it is $d(\kappa^{xy}/T)$ which is a closed 1-form on the parameter space which includes T.

We can try to define the difference of κ_{xy}/T for two materials by integrating the 1-form along a path connecting them.

But what if the result depends on the path? In other words, what if this 1-form is closed but not exact?

What is the topology of the parameter space, anyway?

Energy currents on a lattice

Let $\Lambda \subset \mathbb{R}^d$ be a "lattice", and $H = \sum_{p \in \Lambda} H_p$ be a Hamiltonian. Local Hilbert space are assumed finite dimensional. $[H_p, H_q] = 0$ if $|p - q| > \delta$.

Define the energy current from q to p as $J^E_{pq} = i[H_p, H_q]$. Then one has a lattice conservation equation

$$\frac{dH_p}{dt} = \sum_{q \in \Lambda} J^E_{pq}.$$

Note $J^E_{pq} = -J^E_{qp}$.

Note also that $J^E_{pq} = 0$ if $|p - q| > \delta$, but the energy current does not just flow between nearest neighbors.

Vietoris-Rips chain complex

Fix $\delta > 0$.
A Vietoris-Rips n-chain on Λ is a skew-symmetric function of n points of Λ which vanishes whenever any pairwise distance is greater than δ.

An exponential VR n-chain is defined similarly, but the function decays exponentially whenever the points are far apart.

Boundary operator lowers the degree of a chain:

$$(\partial A)(p_1, \ldots, p_n) = \sum_{p_0 \in \Lambda} A(p_0, p_1, \ldots, p_n).$$

It satisfies $\partial^2 = 0$ thanks to skew-symmetry.

J^E_{pq} is an operator-valued 1-chain in the VR chain complex.

Energy magnetization on a lattice

In a stationary state $\sum_{q\in\Lambda}\langle J_{pq}\rangle = 0$. One expects there exists a skew-symmetric function $M^E_{pqr} : \Lambda\times\Lambda\times\Lambda\to\mathbb{C}$ such that

$$\langle J^E_{pq}\rangle = \sum_r M^E_{pqr}.$$

M^E_{pqr} is a 2-chain such that $\langle J^E\rangle = \partial M^E$. It is defined up to $M^E \mapsto M^E + \partial P$ where P is a 3-chain. One expects M^E_{pqr} to decay rapidly when distances between p, q, r are large.

Kitaev (2006): there is no canonical expression for M^E, but there is a canonical expression for variation of M^E w.r. to parameters of H. Let

$$\mu^E_{pqr} = -\beta\langle\langle DH_p; J^E_{qr}\rangle\rangle + \text{cyclic permutations.}$$

Then

$$D\langle J^E\rangle = \partial\mu^E.$$

Cochains

Fix $\delta > 0$.

An n-cochain is a skew-symmetric function of $n+1$ point of Λ which either grows sub-exponentially when points are far apart, or is defined on some neighborhood of the diagonal.

The evaluaton of a cochain α on a chain A is

$$A(\alpha) = \frac{1}{(n+1)!} \sum_{p_0,\ldots,p_n} A(p_0,\ldots,p_n)\alpha(p_0,\ldots,p_n).$$

Not always defined because of convergence issues. To make well-defined, need to assume that $\alpha(p_0,\ldots,p_n)$ decays when $p_0,\ldots,p_n$ are far from a fixed point on Λ.

Coboundary operator δ is defined via $\partial A(\alpha) = A(\delta\alpha)$. It satisfies $\delta^2 = 0$.

We also have a supercommutative cup product on cochains such that $\delta(\alpha \cup \beta) = \delta\alpha \cup \beta \pm \alpha \cup \delta\beta$.

Hall conductance on a lattice

Consider a 2d lattice system, $H = \sum_p H_p$ with a $U(1)$ symmetry $Q = \sum_p Q_p$ where each Q_p acts only on site p and $[Q, H_p] = 0$.

A convenient formula for Hall conductance where V is already infinite:

$$\sigma_{xy}(T) = \frac{1}{2} \lim_{s \to 0} \beta \int_0^\infty e^{-st} \langle\langle J(t)(\delta f); J(0)(\delta g)\rangle\rangle - (f \leftrightarrow g).$$

Here J_{pq} is the $U(1)$ current on the lattice, $J_{pq} = i[Q_p, H_q] - i[Q_q, H_p]$, and f and g are functions on $\mathbb{R}^2$ which are smeared step-functions in x and y directions, respectively.

Key fact: The above expression does not depend on the choice of f and g provided Kubo pairings of currents decay both in space and time.

Proof uses $[Q_p, Q_q] = 0$.

Thermal Hall conductance on a lattice

A naive generalization to the thermal case is

$$\kappa_{xy}^{Kubo}(T) = \frac{1}{2}\lim_{s\to 0}\beta^2\int_0^\infty e^{-st}\langle\langle J^E(t)(\delta f); J^E(0)(\delta g)\rangle\rangle - (f \leftrightarrow g)$$

It does not work because $[H_p, H_q] \neq 0$ in general. Need a correction from energy magnetization.

Key idea: correct not the naive Kubo formula for κ_{xy} but its derivative w. r. to parameters:

$$D\kappa_{xy}(T) = D\kappa_{xy}^{Kubo}(T) - 2\beta\mu^E(\delta f \cup \delta g).$$

This is a 1-form on the space of parameters.

Phase transitions

Can regard T as one of the parameters, but then have to work with $\Psi = D(\kappa_{xy}/T)$. We regard $T = 0$ as the boundary of the enlarged parameter space.

Correlators and Kubo pairings decay away from phase transitions (1st or 2nd order). The above formulas and statements make sense only away from phase transitions.

Once we remove the locus where phase transitions occur, the topology of the parameter space can become complicated.

It is very plausible that the parameter space is connected, since we are allowed to break all symmetries.

Path-independence of the relative thermal Hall conductance

Given any two points a and b in the enlarged parameter space with finite correlation length and time, we can try to define the difference of κ_{xy}/T for a and b by

$$\int_\Gamma \Psi,$$

where Γ is any path connecting a and b (and avoiding phase transitions). But does it depend on the choice of Γ?

No, Ψ is exact. This follows from the fact that $\mu^E(\delta f \cup \delta g)$ can be written as an exact expression up to exponentially small terms.

A relative invariant of gapped 2d systems

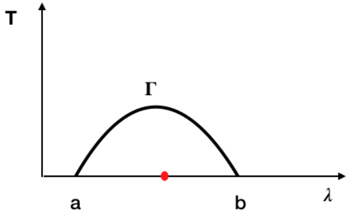
Let a and b be two 2d lattice systems at $T = 0$ with gapped Hamiltonians. Suppose there is a path Γ from a to b. Let

$$\kappa(a, b) = \int_\Gamma \Psi.$$

We showed that $\kappa(a, b)$ is independent of the choice of Γ.

The physical meaning of $\kappa(a, b)$ is the difference of the values of $\lim_{T\to 0}(\kappa_{xy}/T)$ for a and b.

A relative invariant of gapped 2d systems, cont.



We have also argued (non-rigorously):

- $\kappa(a, b)$ is well-defined (integral is convergent)
- $\kappa(a, b)$ does not change as one varies the endpoints a and b provided no phase transition is crossed.
- $\kappa(a, b)$ is equal to $\frac{\pi}{6}(c_L - c_R)$ where $c_L - c_R$ is the chiral central charge of the modes living on the junction between phases a and b

To argue items 1 and 2 we assumed that the limits $T \to 0$ and $V \to \infty$ commute.

Thermal Hall conductance for free fermions

Consider free fermions on a lattice, with a $U(1)$ symmetry. This means that H has the form

$$H = \sum_{p,q} h_{pq} a_p^\dagger a_q,$$

where $a_p, a_p^\dagger$ generate Clifford algebra, and h_{pq} has finite range.

Here there are two invariants: $\sigma_{xy}(0)$ and $\int_\infty^0 \Psi = \lim_{T\to 0}(\kappa_{xy}/T)$.

One can evaluate Ψ explicitly on the path Γ along which only T changes and get

$$\lim_{T\to 0} \frac{\kappa_{xy}}{T} = \frac{\pi^2}{3}\sigma_{xy}(0).$$

This is known as Wiedemann-Franz law (by analogy with the texbook relation between ordinary heat conductivity κ_{xx} and ordinary electric conductivity σ_{xx} which holds in classical theory of transport).

Concluding remarks

- I explained a way to identify bulk correlators in gapped 2d systems which control gravitational anomalies in the edge CFT
- These correlators are deformation invariants of gapped 2d Hamiltonians
- Can be generalized to higher dimensions, providing a way to test the conjecture that cobordisms control gapped phases of lattice systems
- Can be generalized to invariants of families of gapped systems (higher-dimensional generalizations of Berry curvature)
- A more powerful approach is needed to show that for lattice systems the invariants are rational numbers.