Aspects of a Correspondence Between Quivers and Knots

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Why is this striking?

- Two beautiful subjects, both with deep ties to physics.
- The are very different, hint of an interesting relation.

[♠][Kucharski-Reineke-Stosic-Sulkowski '17]

Knot theory distinguishes inequivalent embeddings $K : S^1 \hookrightarrow S^3$ by an assignment of topological invariants.

The HOMFLY-PT polynomial is defined recursively by

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Example

$H_1\Big($  $\Big) = \dfrac{a - a^{-1}}{q - q^{-1}} \left(a^{-2} q^2 - a^{-4} + a^{-2} q^{-2}\right)\,.$

A quiver Q is an oriented graph, with nodes Q_0 connected by arrows. Let C_{ij} be the number of arrows $i \to j$.

A representation $M_{\vec{d}}$ of dimension $\vec{d} \in Q_0\mathbb{N}$ is the assignment

- vector spaces $\mathbb{C}^{d_i}$, $i = 1 \ldots |Q_0|$
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A representation $M_{\vec{d}}$ is stable with respect to $\vec{\theta} \in Q_0\mathbb{R}$ if $\vec{d} \cdot \vec{\theta} = 0$, and $\vec{d'} \cdot \vec{\theta} > 0$ for every proper sub-representation $\vec{d'} \leqslant \vec{d}$.

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We will focus on symmetric quivers: $C_{ij} = C_{ji}$.
Their representation theory is completely understood♠

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$$= \prod_{\vec{d}} \prod_{j\in\mathbb{Z}} (q^j \vec{x}^{\vec{d}}; q^2)_\infty^{(-1)^j \Omega_{\vec{d},j}}$$

$\Omega_{\vec{d},j}$ are positive integers.

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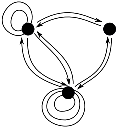
The Knots-Quivers correspondence♠ conjectures that for each K there is a quiver Q, and integers a_i, q_i, such that

$$P^Q(\vec{x}, q)\Big|_{x_i = x\, a^{a_i}\, q^{q_i}} = \sum_{r \geqslant 0} H_r(a, q)\ x^r$$

Example:♣



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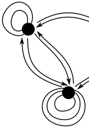
Example:♣  

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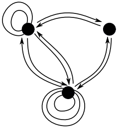

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Evidence: (2,p), (3,p) torus knots; $TK_{2|p|+2}$, TK_{2p+1} twist knots. Proved for rational links.♦

♠[Kucharski-Reineke-Stosic-Sulkowski '17]; ♣For normalized HOMFLY-PT; ♦[Stosic-Wedrich '17]

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Why do knot invariants admit a quiver description?

◦ Given a knot K, how to get the dual quiver Q?

◦ What is the meaning of nodes and arrows of Q?

◦ What is the meaning of parameters a_i, q_i?

◦ Is there a unique quiver Q for a given knot K?

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Strategy: understand connection via String Theory.

Knots in physics

$H_R(a{=}q^N, q^2{=}e^{\frac{2\pi i}{N+k}}) = \langle W_R[K]\rangle_{S^3}$ in $U(N)_k$ Chern-Simons.♠

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Without loops, Chern-Simons theory on S^3 is equivalent to open topological strings with $e^{g_s} = q^2$ on $T^\star S^3$ with N A-branes.[♣]

The 't Hooft limit corresponds to a geometric transition, leading to closed topological strings on the resolved conifold with $t = N g_s$.[♦]

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Knots can be reintroduced by insertion of a "knot conormal" brane on $L_K \subset T^* S^3$, which transitions to a brane in the conifold Y.♥

$$Z_{top}^{open}(Y, L_K) = \sum_{r \geqslant 0} H_r(a,q)\, x^r$$

♠[Witten '89]; ♣ [Witten '95]; ♦ [Gopakumar-Vafa '98]; ♥ [Ooguri-Vafa '99]

x is a brane modulus for $L_K \simeq \mathbb{R}^2 \times S^1$: one real deformation♠ complexified by $U(1)$ holonomy

$$x = e^{i\,X} = e^{r + i \oint_{S^1} A} \qquad \text{(longitude)}$$

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Classical phase space of flat $U(1)$ connections on $\partial L_K \simeq T^2$ is also T^2. Canonical quantization yields plane waves[♣]

$$\psi_n(x) = e^{\frac{i}{\hbar}\, X \cdot P_n}\,, \qquad\qquad P_n = n\,\hbar\,.$$

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$$\psi_n(x) = e^{\frac{i}{\hbar}\,X\cdot P_n}\,, \qquad\qquad P_n = n\,\hbar\,.$$

Therefore $x^n = \psi_n(x)$ is a wavefunction, and so is

$$Z_{top}^{open}(Y, L_K) = \sum_{r\geqslant 0} H_r(a,q)\,\psi_r(x) \quad \in \; \mathscr{H}[\partial L_K]$$

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Let's consider the semiclassical limit $(q^2 = e^{g_s} \to 1)$ of both sides:

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$$\sum_{n \geqslant 0} H_n(a,q)\, x^n \ \sim \ \int \frac{dy}{y}\, e^{\,\frac{1}{g_s}\left[-\widetilde{\mathcal{W}}_{L_K}(a,y) \ + \ \log x \cdot \log y\right] + \ldots}$$

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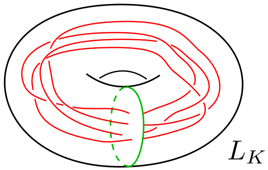
$$\begin{array}{ccc} Z_{top}^{open}(a,q,x) & \leftarrow \text{[Fourier]} \rightarrow & H_n(a,q) \\ \downarrow & & \downarrow \\ W_{Disk}(a,x) & \leftarrow \text{[Legendre]} \rightarrow & \widetilde{\mathcal{W}}_{L_K}(a,y) \end{array}$$

The Gromov-Witten disk potential is related by Legendre transform to the source potential of a $U(1)$ Chern-Simons theory on L_K.

Sources in U(1) Chern-Simons on L_K arise from boundaries of holomorphic disks wrapping $S^1 \sim K$.



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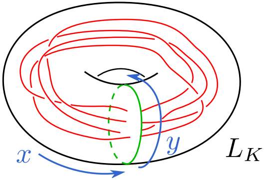

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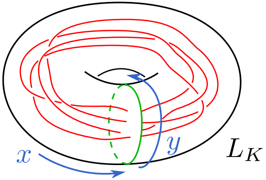

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$$\exp\left(\frac{\partial \widetilde{\mathcal{W}}_{L_K}}{\partial \log y} - \log x\right) = 1 \quad \leftrightarrow \quad \exp\left(\frac{\partial W_{Disk}}{\partial \log x} - \log y\right) = 1$$

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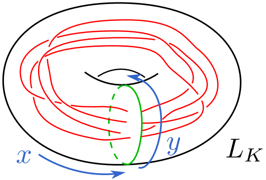

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$$\leftrightarrow \quad A(x, y, a) = 0 \quad \subset \quad \mathbb{C}_x^* \times \mathbb{C}_y^*$$

recovering the Abel-Jacobi map on the augmentation curve.♠

♠[Ng'10]; [Ekholm-Etnyre-Ng-Sullivan'10]; [Aganagic-Vafa '01, '12]; [AENV'13]

Is there a notion of semiclassical limit for quivers?

$$P^Q(\vec{x}, q) = \sum_{d_1 \ldots d_m \geqslant 0} (-q)^{\vec{d} \cdot C \cdot \vec{d}} \prod_{i=1}^{m} \frac{x_i^{d_i}}{(q^2; q^2)_{d_i}}$$

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We <u>define</u> such a limit by setting $y_i = \lim_{g_s\to 0} q^{d_i}$

$$P^Q(\vec{x},q) \sim \int \prod_i \frac{dy_i}{y_i} e^{\frac{1}{g_s}\left(\underbrace{\sum_{i=1}^{|Q_0|} \mathrm{Li}_2(y_i) + C_{ij}\log y_i \log y_j}_{\widetilde{\mathcal{W}}_Q} + \log x_i \cdot \log y_i\right)} + \dots$$

Now $\widetilde{\mathcal{W}}_Q$ encodes a <u>finite</u> set of sources (one for each node in Q), together with a <u>finite</u> set of couplings (C_{ij} counts arrows $i \to j$).

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In sharp contrast with the much more complicated $\widetilde{\mathcal{W}}_{L_K}$.

To interpret this, we consider Legendrian constraints (saddles)

$$A_i(x_i, y_i) := 1 - y_i - x_i \prod_j y_j^{C_{ij}} = 0$$

defining "quiver A-polynomials".

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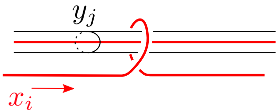
Since $x_i \sim x$ it follows that $\prod y_i = y$. This suggests that

- each source winds once around S^1
- y_i is the contribution of a source to the meridian on ∂L_K
- C_{ij} are linking numbers: meridians shift longitudes



x_i, y_i are holonomies on a tubular neighbourhood around the i-th source. This enlarges the phase space to

$$\mathcal{M}_Q = (\mathbb{C}^* \times \mathbb{C}^*)^{|Q_0|}$$

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Quiver A-polynomials $A_i(x_i, \vec{y}) = 0$ define a Lagrangian $\mathcal{L}_Q \subset \mathcal{M}_Q$ of complex dimension $|Q_0|$.

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The role of the KQ change of variables $x_i = x\, a^{a_i}\, q^{q_i}$ is to carve out a 1-dimensional sub-variety: $A(x, y, a) = 0$.

It is determined by the embedding of $L_K \hookrightarrow Y$, since a_i encode wrappings of basic disks around the $\mathbb{P}^1$ in Y.

Back to holomorphic disks:

$$\Big[\text{U(1) Chern-Simons on } L_K\Big] \quad \text{vs} \quad \Big[\text{topological strings on } (Y, L_K)\Big]$$

$$\widetilde{\mathcal{W}}_{L_K}(a,y) \quad \leftarrow \text{[Legendre]} \rightarrow \quad W_{Disk}(a,x)$$

$$\uparrow$$

(dual)

$$\downarrow$$

(finite!) $\rightsquigarrow$ $\widetilde{\mathcal{W}}_Q(y_i)$

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$$\begin{tikzcd}
& \widetilde{\mathcal{W}}_{L_K}(a,y) \arrow[r, leftrightarrow, "\text{[Legendre]}"] \arrow[d, leftrightarrow, "\text{(dual)}"] & W_{Disk}(a,x) \arrow[d, leftrightarrow, "(x_i \,=\, x a^{a_i} q^{q_i})"] \\
\text{(finite!)} \arrow[r, rightsquigarrow] & \widetilde{\mathcal{W}}_Q(y_i) \arrow[r, leftrightarrow, "\text{[Legendre]}"] & W_Q(x_i)
\end{tikzcd}$$

The quiver disk potential W_Q refines W_{Disk} by $\vec{d}$-grading

$$W_{Disk} = \sum_{r,i,j} (-N^K_{r,i,j}) \operatorname{Li}_2(x^r a^i) \qquad \text{LMOV}$$

$$W_Q = \sum_{\vec{d},j} (-1)^{|\vec{d}|+j} \Omega_{\vec{d},j} \operatorname{Li}_2(\vec{x}^{\vec{d}}) \qquad \text{DT}$$

Quiver description of the spectrum of holomorphic curves

Basic disks

- one for each node
- $x_i \sim x a^{a_i}$: wrap once around K and a_i times around $\mathbb{P}^1$

Boundstate disks

- stable Q-rep. contains $\vec{d} = (\dots d_i \dots)$ copies of basic disks
- counted by $\Omega_{\vec{d},j}$
- completely fixed by linking numbers C_{ij}

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Higher genus curves

- are counted by P^Q
- are generated from basic disks too, by quiver dynamics!

Back to the main question

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What is the origin of “quiver dynamics”?

Embedding open topological strings into M theory

open topological string		M theory	
	Y		$Y \times S^1 \times \mathbb{R}^4$
A-brane on	L_K	M5 on	$L_K \times S^1 \times \mathbb{R}^2$
instanton	$[\beta] \in H_2^{rel}(Y, L_K)$	M2 on	$\beta \times S^1$

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M5 engineers a 3d $\mathcal{N} = 2$ theory $T[L_K]$ on $S^1 \times \mathbb{R}^2$.
Its (K-theoretic) vortex partition function counts M2 branes.♠

$$Z_{top}^{open}(Y, L_K) = Z_{vortex}(T[L_K])$$

♠[Dimofte-Gukov-Hollands '10]

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	Y		$Y \times S^1 \times \mathbb{R}^4$
A-brane on	L_K	M5 on	$L_K \times S^1 \times \mathbb{R}^2$
instanton	$[\beta] \in H_2^{rel}(Y, L_K)$	M2 on	$\beta \times S^1$

M5 engineers a 3d $\mathcal{N} = 2$ theory $T[L_K]$ on $S^1 \times \mathbb{R}^2$.
Its (K-theoretic) vortex partition function counts M2 branes.♠

$$Z_{top}^{open}(Y, L_K) = Z_{vortex}(T[L_K])$$

Think of the quiver as describing the dynamics of either:

- M2 branes with linking boundaries
- BPS vortices of $T[L_K]$

♠[Dimofte-Gukov-Hollands '10]

To study vortex dynamics, we need to understand $T[L_K]$:

$$Z_{vortex}(T[L_K]) \sim \int \frac{dy}{y}\, e^{\frac{1}{g_s}[-\widetilde{\mathcal{W}}_{L_K}(a,y) + \log x \cdot \log y] + \ldots}$$

An IR description can be obtained via a 3d-3d dictionary♠

- $\circ$ $\int \frac{dy}{y}$: $U(1)$ gauge symmetry
- $\circ$ $\mathrm{Li}_2(e^{\mu} y^{Q}) \subset \widetilde{\mathcal{W}}_{L_K}$: 1-loop of a chiral with charge Q, mass μ
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But this description is not very useful: a $U(1)$ gauge theory with complicated matter spectrum and interactions ($\widetilde{\mathcal{W}}_{L_K}$ is infinite).

♠[Dimofte-Gaiotto-Gukov'09; Fuji-Gukov-Sulkowski'13;...]

However we have a dual description of $Z_{vortex}(T[L_K])$

$$P^Q(\vec{x},q) \sim \int \prod_i \frac{dy_i}{y_i}\; e^{\frac{1}{g_s}\left(-\widetilde{\mathcal{W}}_Q + \log x_i \cdot \log y_i\right) + \ldots}$$

specialized to $x_i = x\, a^{a_i}$, with $\widetilde{\mathcal{W}}_Q = \sum_i \mathrm{Li}_2(y_i) + C_{ij} \log y_i \log y_j$.

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This defines a dual theory $T[Q]$

- gauge group $U(1)_1 \times \cdots \times U(1)_{|Q_0|}$
- matter: one chiral for each node, with charge δ_{ij} under $U(1)_j$
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The origins of the Knots-Quivers correspondence can be traced to a quantum mechanics of <u>BPS vortices in $T[Q]$</u>.♠

♠Admitting a quiver description [Hwang-Yi-Yoshida '17].

Quiver quantum mechanics from the viewpoint of M2 branes

- nodes: M2 wrapping basic holomorphic disks
- links: bifundamental light modes at M2-M2 intersections

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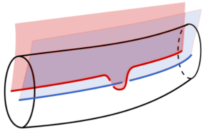
Answer from Knot Contact Homology: "standardize" discs, by stretching along certain submanifolds♠

- Morse function $f : L_K \to \mathbb{R}$ with absolute minimum on the zero-section of $\mathbb{R}^2 \to L_K \to S^1$; let D_0 be its disc fiber
- Given β_i with $\partial\beta_i \subset L_K$ define $\sigma_i' = \bigcup\{\text{flow lines of } \nabla f\}$
- At infinity $\sigma_i' \to m\lambda + n\mu \in H_1(\partial L_K, \mathbb{Z})$
- Standardize by defining $\sigma_i = \sigma_i' - nD_0$

♠[Ekholm-Ng'18]

Linking number (M2-M2 intersections)



$$C_{ij} = \mathrm{lk}(\partial\beta_i, \partial\beta_j) = \partial\beta_i \cdot \sigma_j = \sigma_i \cdot \partial\beta_j$$

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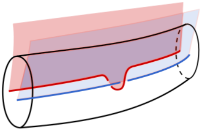

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We can also define a notion of self-linking:

- introduce the 4-chain $C = \bigcup\{\text{flow lines of } J\nabla f\}$ in Y
- choose a "pushoff" vector field ν along $\partial\beta$

$$\mathrm{slk}(\beta) = \underbrace{\partial\beta_\nu \cdot \sigma_\beta}_{m\cdot n \equiv C_{ii}} - \beta_{J\nu} \cdot C$$

Framing

Both $\sum_r H_r(a,q)x^r$ and Z_{top}^{open} depend on a choice of $f \in \mathbb{Z}$.

- In knot theory, it's an ambiguity arising in point-splitting regularization of Chern-Simons.
- In open topological strings, it's a semiclassical ambiguity of which cycle on ∂L_K is the "real longitude".

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But its effects are well understood: e.g.

$$A(x,y;a) = 0 \quad \to \quad A(x \cdot y^f, y; a) = 0$$

leads to highly nontrivial changes in the disk potential

$$W_{Disk}(a,x) = \int \log y_\star \, d\log x = \sum (-N_{r,i,j}^K) \, \mathrm{Li}_2(x^r a^i)$$

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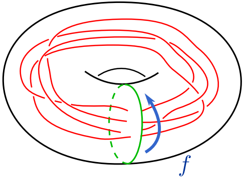
What happens on the quiver side?

Since x, y are meridian and longitude on ∂L_K, geometrically $x \to x \cdot y^f$ corresponds to performing f Dehn twists



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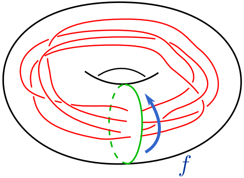

Therefore we expect

- an overall shift of linking numbers $C_{ij} \to C_{ij} + f$
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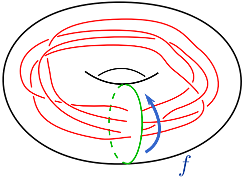

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It also implies that Q is not unique, in a way that is under control.

♠[Kucharski-Reineke-Stosic-Sulkowski '17]

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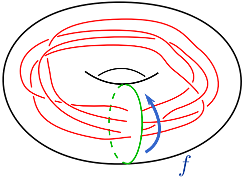

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On $T[Q]$, framing acts by overall shift of Chern-Simons couplings.♣

♠[Kucharski-Reineke-Stosic-Sulkowski '17]; ♣[Witten '03]

Conclusions

- The quiver description of knot invariants originates from the dynamics of BPS vortices of a 3d $\mathcal{N}=2$ theory $T[Q]$.

- The structure of $T[Q]$ is encoded by the quiver

  - gauge group $U(1)_1 \times \cdots \times U(1)_{|Q_0|}$
  - one charged chiral for each $U(1)$
  - mixed Chern-Simons couplings C_{ij}
  - Fayet-Iliopoulos terms $\log x_i = \log x\, a^{a_i}$.

- Augmentation polynomials of knots admit a decomposition into universal blocks, the "Quiver A-polynomials" A_i

- Quivers encode counts of holomorphic curves on (Y, L_K)
  - a basic holomorphic disk on each node
  - interactions encoded by linking of disk boundaries
  - through quiver QM, disks generate all higher-genus curves too!

- The knot-quivers correspondence leads to a detailed dictionary between its geometric and physical interpretations

Quiver	Geometry	Physics
node i	basic holomorphic disk β_i	M2 brane / BPS vortex
edges C_{ij}	$\mathrm{lk}(\partial\beta_i, \partial\beta_j)$	M2-M2 intersection / CS cplg.
x, y	holonomies on $\partial L_K \simeq T^2$	moduli for $T[L_K]$
x_i, y_i	holonomies on $(T^2)_i \subset \partial(L_K \backslash \{\partial\beta_j\})$	moduli for $T[Q]$
a_i	wrappings of $\mathbb{P}^1$	flavor charge of $U(1)_a$
q_i	self-linking $\mathrm{slk}(\beta_i)$	spin $SO(2) \circlearrowright \mathbb{R}^2$
$C_{ij} \to C_{ij} + f$	(Dehn twist)^f	overall shift of CS couplings

Thank You.