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Magnificent Four with Colors

NIKITA NEKRASOV

Uppsala, July 3, 2019

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Magnificent Four with Colors, and Beyond (?) Eleven Dimensions

Nikita Nekrasov

Simons Center for Geometry and Physics

Strings-Math'19 Uppsala July 3

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A popular approach to quantum gravity

is to approximate the space-time geometry by some discrete structure

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A popular approach to quantum gravity

is to approximate the space-time geometry by some discrete structure

Then develop tools for summing over these discrete structures

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A popular approach to quantum gravity

is to approximate the space-time geometry by some discrete structure

Then develop tools for summing over these discrete structures

Tuning the parameters so as to get, in some limit

Smooth geometries

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To some extent

two dimensional quantum gravity

is successfully solved in this fashion

using matrix models

$$\log \int_{N\times N} dM\, e^{-N \operatorname{tr} V(M)} \sim \sum_{\text{fat graphs} \leftrightarrow \text{triangulated Riemann surfaces}}$$

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To some extent

two dimensional quantum gravity

is successfully solved in this fashion

using matrix models

$$\log \int_{N\times N} dM\, e^{-N \mathrm{tr} V(M)} \sim \sum_{g=0}^{\infty} N^{2-2g} \sum_{\text{genus } g \ \textit{Riemann surfaces}}$$

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Going up in dimension

proves difficult

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Going up in dimension

proves difficult – the obvious generalization of a matrix model

is the so-called tensor theory

$$M_{ij} \longrightarrow \Phi_{ijk}$$

There is no analogue of genus expansion for general three-manifolds

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Going up in dimension

proves difficult – the obvious generalization of a matrix model

is the so-called tensor theory

$$M_{ij} \longrightarrow \Phi_{ijk}$$

There is no analogue of genus expansion for general three-manifolds

However an interesting large N scaling has been recently found

In the context of the SYK model, $g \mapsto$ Gurau index

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In this talk

Random three dimensional geometries

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In this talk

I do not claim to quantize three dimensional Einstein gravity

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In this talk

Models of random three dimensional geometries, from which

we may learn about eleven dimensional super-gravity/M-theory

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In this talk

Models of random three dimensional geometries, from which

we may learn about eleven dimensional super-gravity/M-theory

and beyond

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Geometries from partitions

One way to generate a d-dimensional random geometry

Is from some local growth model in $d+1$-dimensions

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Geometries from partitions

For example, start with the simplest “mathematical” problem

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Geometries from partitions

For example, start with the simplest problem: counting natural numbers

$$1,\ 2,\ 3, \ldots$$

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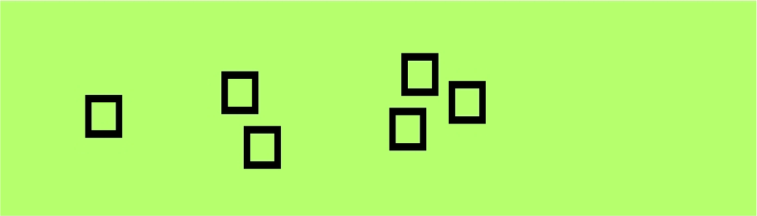
Geometries from partitions

For example, start with the simplest problem: accounting

$$1,\ 2,\ 3, \ldots$$



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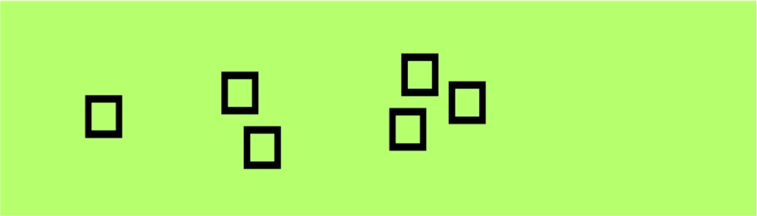
Geometries from partitions

Accounting for objects without structure

$1,\ 2,\ 3, \dots$



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Geometries from partitions

Now add the simplest structure: partitions of integers

$$(1)\,;\;\;(2)\,,\;\;(1,1)\,;\;\;(3)\,,\;\;(2,1)\,,\;\;(1,1,1)\,;\;\;\ldots$$

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Geometries from partitions

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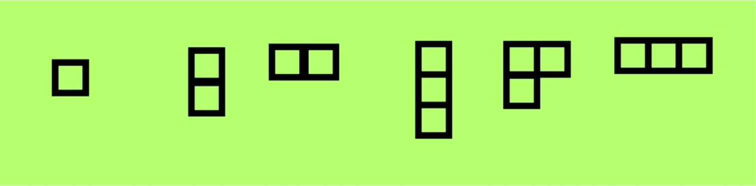
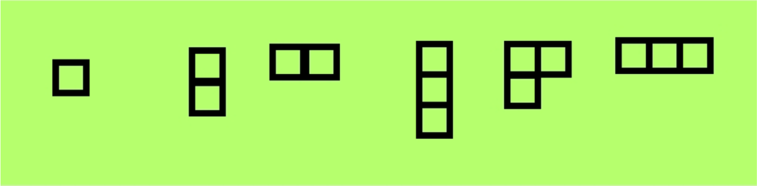


Geometries from partitions

The structure: partitions of integers as bound states

$$(1)\,; \qquad (2)\,,\ (1,1)\,; \qquad (3)\,,\ (2,1)\,,\ (1,1,1)\,;\ \ldots$$



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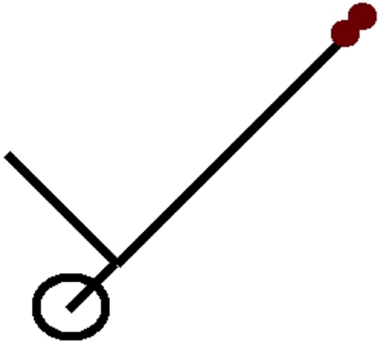
Partitions as growth model

Gardening and bricks



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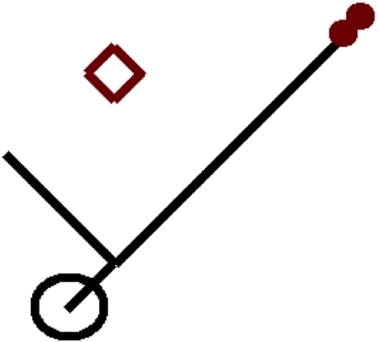

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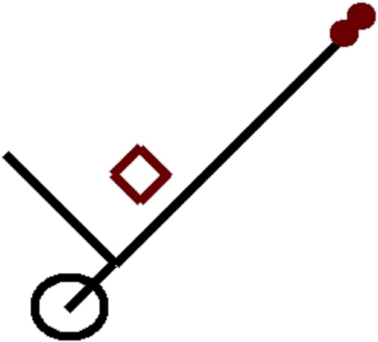

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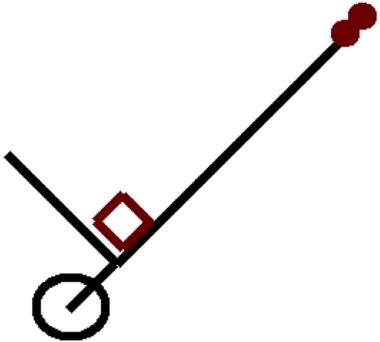

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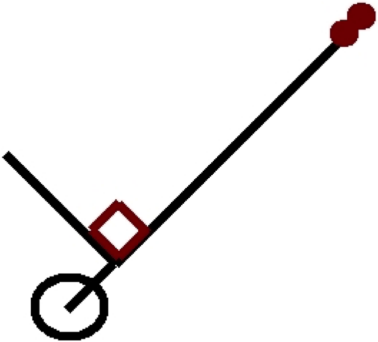
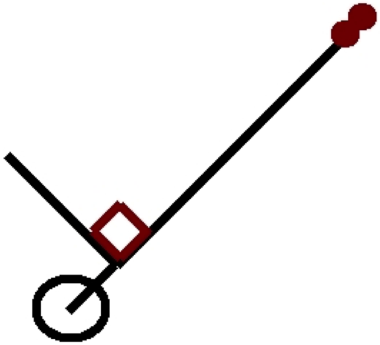


Partitions as growth model

Partition (1) made of brick



Partitions as growth model

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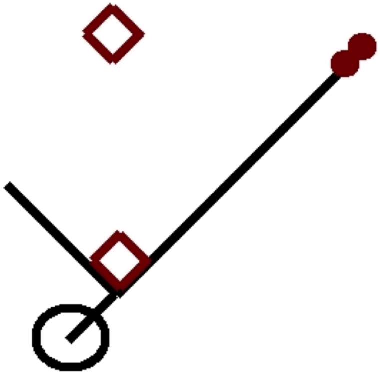

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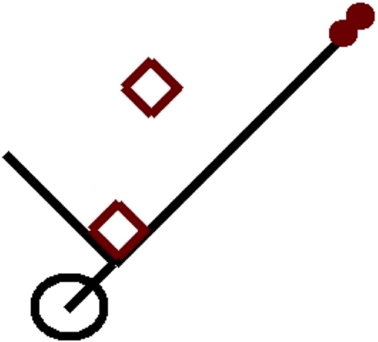

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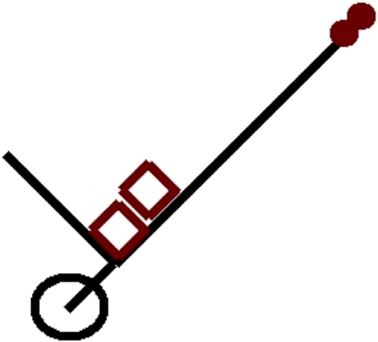

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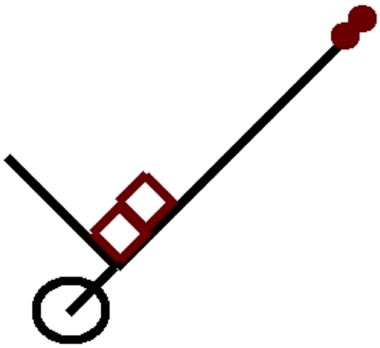
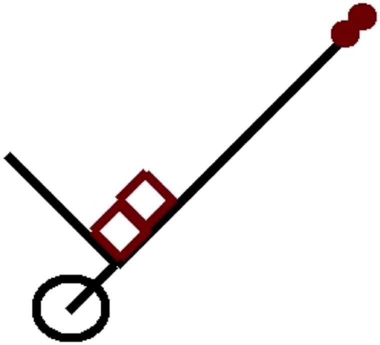
Gardening and bricks



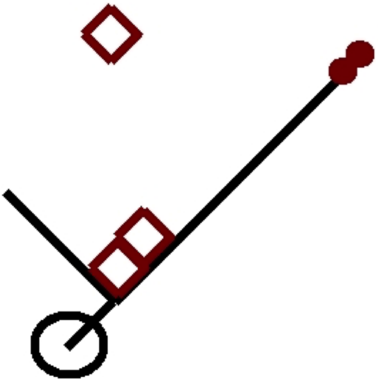
Partitions as a growth model

Partition (2) made of bricks



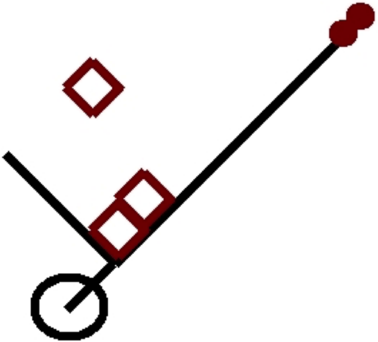
Partitions as a growth model

Gardening and bricks



Partitions as a growth model

Gardening and bricks



Partitions as a growth model

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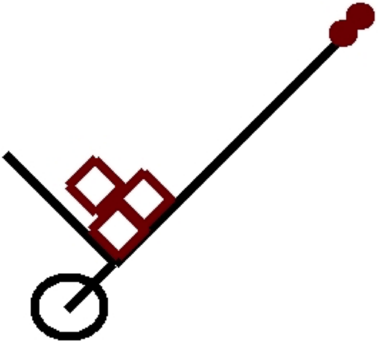
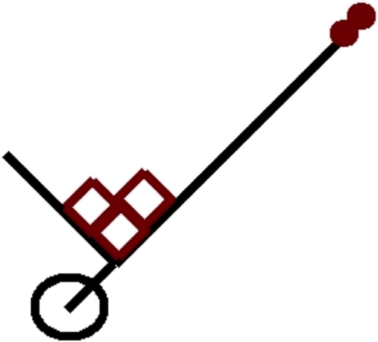


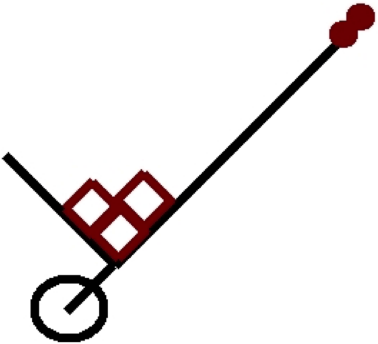
Partitions as a growth model

Gardening and bricks



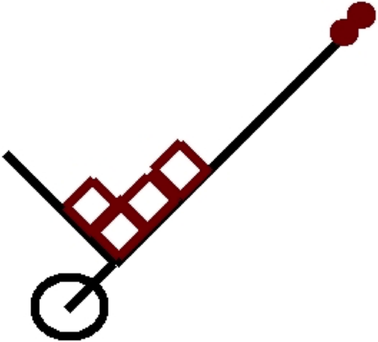
Partitions as a growth model

Partition $(2,1)$ made of bricks



Partitions as a growth model

Partition $(3, 1)$ made of bricks



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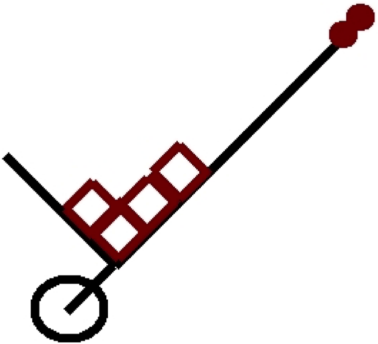
Partitions as a growth model

The probability of a given partition, e.g. $(3, 1)$, is determined by the equality



of the chances of jumps from one partition
e.g. from $(3, 1)$ to another, e.g. $(3, 2)$ or $(4, 1)$, or $(3, 1, 1)$

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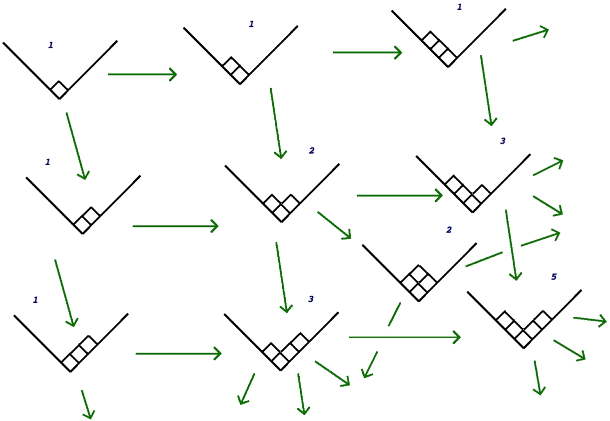
Possibilities of growth: Young graph



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Partitions as a growth model

Thus the probability p_λ of a given partition λ

is proportional to the # of ways it can be built out of the nothing

times the # of ways it can be reduced to nothing

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Partitions as a growth model

Thus the probability p_λ of a given partition λ

is proportional to the # of ways it can be built out of the nothing

times the # of ways it can be reduced to nothing: quantum bricks

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Plancherel measure: symmetry factors

One can calculate this to be equal to

$$p_\lambda = \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 \Lambda^{2|\lambda|} e^{-\Lambda^2}$$

$$= \; e^{-\Lambda^2} \; \left(\prod_{\square \in \lambda} \frac{\Lambda}{\text{hook} - \text{length of } \square} \right)^2$$

For example, $p_{3,1} = \frac{1}{1^2 2^2 4^2 1^2} = \frac{1}{64}$

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Supersymmetric gauge theory

Remarkably, p_λ is the simplest example

of an instanton measure

$$p_\lambda = (\mathrm{sDet}\Delta_{A_\lambda})^{-\frac{1}{2}}$$

i.e. the one-loop (exact) contribution of an instanton $A = A_\lambda$

in $\mathcal{N} = 2$ supersymmetric gauge theory

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Supersymmetric gauge theory and random partitions

Consider $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions

The fields of a vector multiplet are
$A_m,\ m = 1, 2, 3, 4;\ \lambda_{\alpha i},\ \alpha = 1, 2$ and $i = 1, 2;\ \phi, \bar{\phi}$

with the supersymmetry transformations, schematically

$$\delta A \sim \lambda + \bar{\lambda}\,, \qquad \delta\phi \sim \lambda\,, \qquad \delta\bar{\phi} \sim \bar{\lambda}$$

$$\delta(\lambda, \bar{\lambda}) \sim (F^{+} + D_A\phi, F^{-} + D_A\bar{\phi}) + [\phi, \bar{\phi}]$$

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Supersymmetric gauge theory and random partitions

Supersymmetric partition function of the theory can be computed exactly

by localizing on the δ-invariant field configurations, i.e. $F_A^+ = 0$

$$Z = \sum_k \Lambda^{2Nk} \int_{\mathcal{M}_k^+} \text{instanton measure}$$

of some effective measure, including the regularization factors

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Supersymmetric gauge theory and random partitions

The integral over the moduli space can be further simplified by

by deforming the supersymmetry using the rotational symmetry of $\mathbb{R}^4$

$$Z = \sum_k \Lambda^{2Nk} \sum_{\lambda, |\lambda|=k} p_\lambda$$

The deformed path integral is computed by exact saddle point analysis

with λ enumerating the saddle points

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Supersymmetric gauge theory and random partitions

Generic rotation of $\mathbb{R}^4$: $g_{\mathrm{rot}} = \exp \begin{pmatrix} 0 & \varepsilon_1 & 0 & 0 \\ -\varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \\ 0 & 0 & -\varepsilon_2 & 0 \end{pmatrix}$

$$Z = \sum_k \Lambda^{2Nk} \sum_{\lambda,\, |\lambda|=k} p_\lambda(\varepsilon_1, \varepsilon_2)$$

The deformed path integral is computed by exact saddle point

Exact saddle point approximation

for $U(N)$ gauge theory: $\lambda =$ an N-tuple of partitions $\lambda^{(1)}, \ldots, \lambda^{(N)}$

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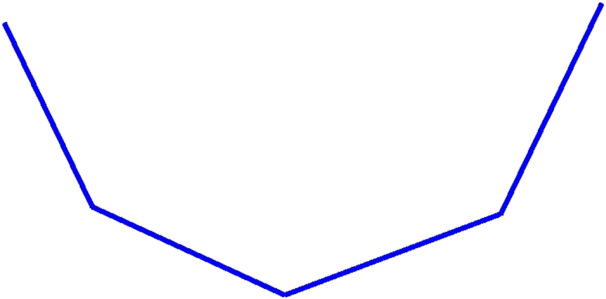
Supersymmetric gauge theory and random partitions

In this way supersymmetric gauge theory becomes a model of

random partitions = random piecewise linear geometries



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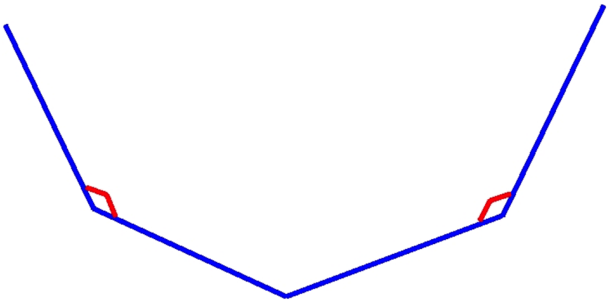

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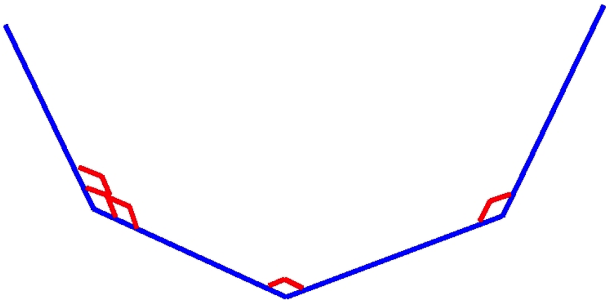

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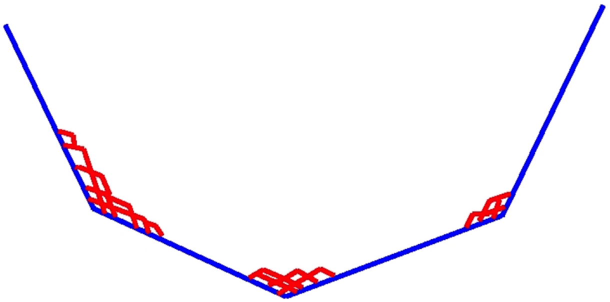

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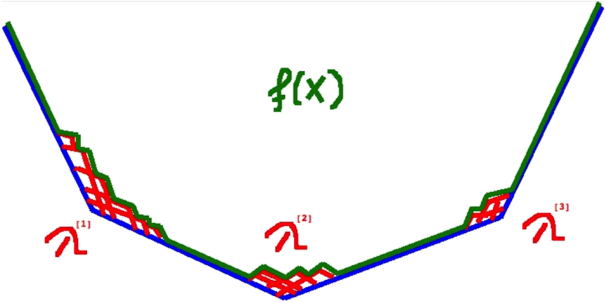

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Supersymmetric gauge theory and random partitions

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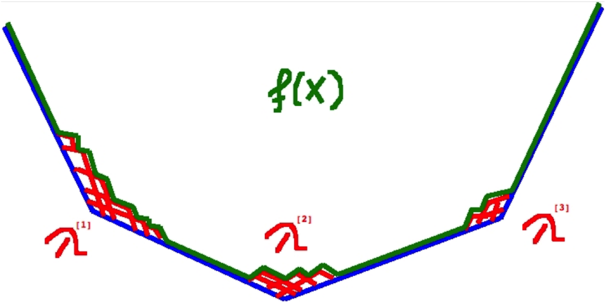

$$p_{\lambda}(\varepsilon_1, \varepsilon_2) = \exp \int \int dx_1 dx_2 \; f''(x_1) f''(x_2) K(x_1 - x_2; \varepsilon_1, \varepsilon_2)$$

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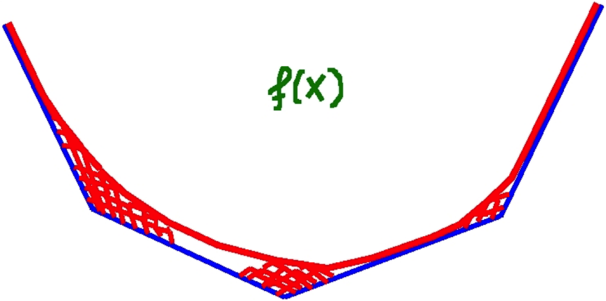
Emergent spacetime geometry

In the limit $\varepsilon_1, \varepsilon_2 \to 0$ (back to flat space supersymmetry)

The sum over random partitions is dominated by the so-called limit shape



$$p_\lambda(\varepsilon_1, \varepsilon_2) \sim \exp \frac{1}{\varepsilon_1 \varepsilon_2} F_\lambda$$

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Higher dimensional gauge theories

The analogous supersymmetric partition functions

can be defined for $d = 4, 5, 6, 7, 8, 9$ dimensional gauge theories

using embedding in string theory for $d > 4$

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Extra dimension

These computations can be used to test some

of the most outstanding predictions of mid-90s, e.g. that

sum over the $D0$-branes = lift to one higher dimension

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Extra dimension

E.g. the max susy gauge theory in $4+1$ dim's

$$Z_{4+1}^{N=1} = \mathrm{Tr}_{\mathcal{H}_{\mathbb{R}^4}}\, g_{\mathrm{rot}}\; g_{\mathrm{R-sym}}\; g_{\mathrm{flavor}}\, (-1)^F$$

$$= \exp \sum_{k=1}^{\infty} \frac{1}{k} F_5(q_1^k, q_2^k, \mu^k, p^k)$$

$$\text{Free energy } F_5(q_1, q_2, \mu, p) = \frac{p}{1-p} \frac{(1-\mu q_1)(1-\mu q_2)}{\mu(1-q_1)(1-q_2)}$$

$q_1 = e^{\mathrm{i}\beta\varepsilon_1}$, $q_2 = e^{\mathrm{i}\beta\varepsilon_2}$, $\beta\varepsilon_1$, $\beta\varepsilon_2$ are the angles of the spatial $\mathbb{R}^4$ rotation

$\mu = e^{\mathrm{i}\beta m}$, m is the mass of the adjoint hypermultiplet,

β is the circumference of the temporal circle

p is the fugacity for the # of instantons =

of $D0$ branes bound to a $D4$ brane in the IIA string picture

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Extra dimension

Remarkably,

$$Z_{4+1}^{N=1}(q_1, q_2, \mu, p) \;=\; \exp \sum_{k=1}^{\infty} \frac{1}{k} F_5\left((\cdot)^k\right) =$$

$=$ Partition function of a minimal $d = 6$, $\mathcal{N} = (0, 2)$ multiplet

On space-time $\mathbb{R}^4 \tilde{\times} \mathbb{T}^2$

$$p = e^{2\pi i \tau}\,, \qquad \tau = \text{complex modulus of the} \quad \mathbb{T}^2$$

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Extra dimension

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On space-time $\mathbb{R}^4 \widetilde{\times} \mathbb{T}^2$

$$p = e^{2\pi i \tau}, \qquad \tau = \text{complex modulus of the } \quad \mathbb{T}^2$$

In agreement with $D4$ brane $=$ $M5$ brane on S^1

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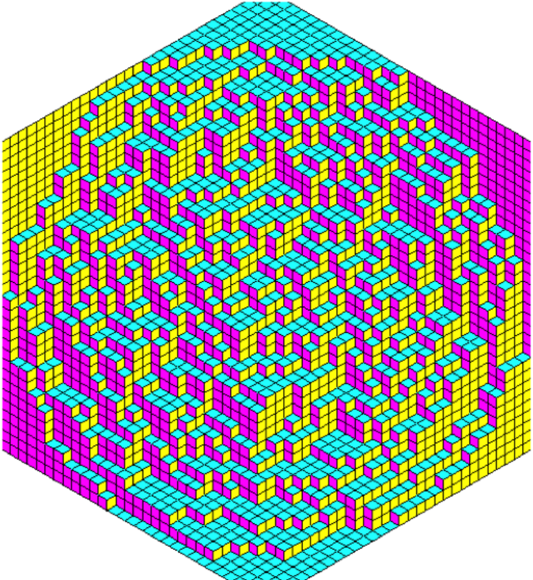
Even higher dimensions: $6+1$

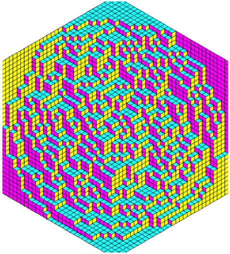
SYM in $6+1$ dim's – $\mathrm{Tr}(-1)^F g$ is expressed as a sum over plane partitions



Even higher dimensions: $6+1$

SYM in $6+1$ dim's – $\mathrm{Tr}(-1)^F g$ is expressed as a sum over plane partitions

$$g_{\mathrm{rot}} = \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{pmatrix}, \; R_i = \exp \mathrm{i}\beta \begin{pmatrix} 0 & \varepsilon_i \\ -\varepsilon_i & 0 \end{pmatrix}$$



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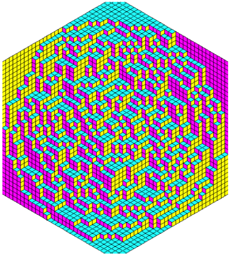
Even higher dimensions: $6+1$

SYM in $6+1$ dim's – $\mathrm{Tr}(-1)^F g$ is expressed as a sum over plane partitions

$$Z_{6+1}^{N=1} = \exp \sum_{k=1}^{\infty} \frac{1}{k} F_7(q_1^k, q_2^k, q_3^k, p^k)$$

Again, p counts instantons = $D0$ branes bound to a $D6$



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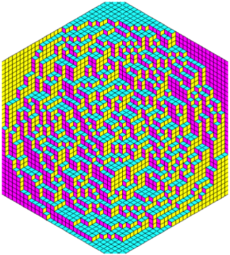
Even more higher dimensions: $6+1 \to 10+1$

It turns out, that the supersymmetric free energy of plane partitions

$$F_7(q_1,q_2,q_3,p) = \frac{\sum_{a=1}^{5}\left(Q_a - Q_a^{-1}\right)}{\prod_{a=1}^{5}\left(Q_a^{\frac{1}{2}} - Q_a^{-\frac{1}{2}}\right)}$$

$$Q_1 = q_1,\, Q_2 = q_2,\, Q_3 = q_3,\, Q_4 = p(q_1q_2q_3)^{-\frac{1}{2}},\, Q_5 = p^{-1}(q_1q_2q_3)^{-\frac{1}{2}}$$



$S(3)$-symmetry enhanced to $S(5)$ symmetry

Twisted Witten index of 11d supergravity!

Plane partitions = 3d Young diagrams

know about (super)gravity in $10+1$ dimensions!

In agreement with: $D6 \to Taub - Nut \approx \mathbb{R}^4$,

$IIA \to$ M-theory

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From 2d and 3d to 4d Young diagrams

Eight dimensional analogue of the ADHM construction

Three complex Hermitian vector spaces are involved: N, M, K

Matrices: $B_a : K \to K$, $a = 1, \ldots, 4$, $I : N \to K$, $\Upsilon : M \to K$

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Eight dimensional analogue of the ADHM construction

Three complex Hermitian vector spaces are involved: N, M, K

Dimensions: $\mathrm{dim} K = k$, $\mathrm{dim} N = \mathrm{dim} M = n$
Matrices: $B_a : K \to K$, $a = 1, \ldots, 4$, $I : N \to K$, $\Upsilon : M \to K$

Υ is a fermion

Equations:

$$[B_1, B_2] + [B_3, B_4]^\dagger = 0 \qquad \text{and cyclic permutations}$$

$$\sum_{a=1}^{4} [B_a, B_a^\dagger] + II^\dagger = r \cdot 1_K$$

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Eight dimensional analogue of the ADHM construction

Three complex Hermitian vector spaces are involved: N, M, K

Matrices: $B_a : K \to K$, $a = 1, \ldots, 4$, $I : N \to K$, $\Upsilon : M \to K$

Symmetry:

$$(B_a) \mapsto (g_{a\dot{b}} B_b) \ , g \in SU(4)$$

$$\Upsilon \mapsto \Upsilon \mathbf{b}^{-1} \ , \mathbf{b} \in U(M)$$

$$I \mapsto I \mathbf{a}^{-1} \ , \mathbf{a} \in U(N)$$

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Eight dimensional ADHM quantum mechanics

Make matrices time-dependent

Supersymmetric Lagrangian

Equations squared = potential term

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From 2d and 3d to 4d Young diagrams

Twisted Witten index = a count of solid n-colored partitions

= 4d Young diagrams

How to visualize them?

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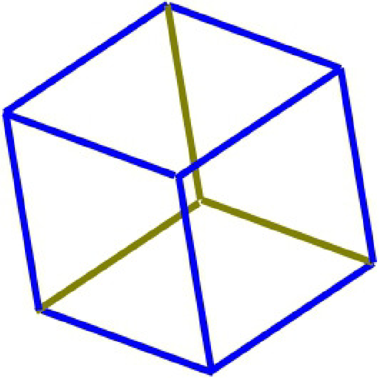
From 2d and 3d to 4d Young diagrams

How to visualize 4d Young diagrams?

Use the projection from $\mathbb{R}^4 \to \mathbb{R}^3$ along the $(1,1,1,1)$ axis



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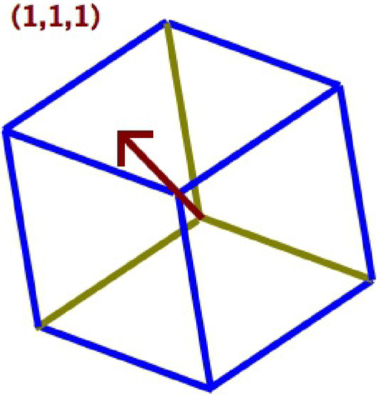
From 2d and 3d to 4d Young diagrams

How to visualize 4d Young diagrams?

Just like the projection from $\mathbb{R}^3 \to \mathbb{R}^2$ along the $(1,1,1)$ axis



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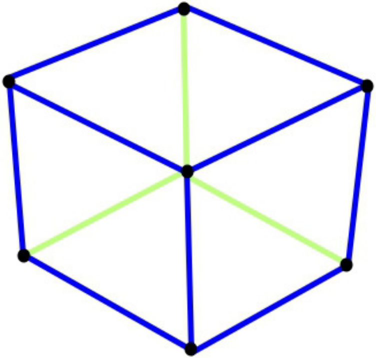
From 2d and 3d to 4d Young diagrams

How to visualize 4d Young diagrams?

The projection from $\mathbb{R}^3 \to \mathbb{R}^2$ gives the tesselation of $\mathbb{R}^2$



By rombi of three orientations

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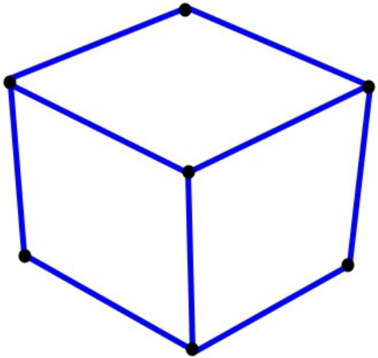
From 2d and 3d to 4d Young diagrams

How to visualize 4d Young diagrams?

The projection from $\mathbb{R}^3 \to \mathbb{R}^2$ gives the tesselation of $\mathbb{R}^2$



By rombi of three orientations

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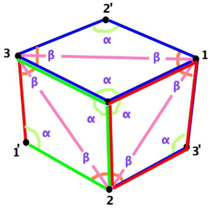
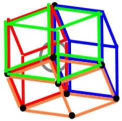
From 2d and 3d to 4d Young diagrams

Projection from $\mathbb{R}^4 \to \mathbb{R}^3$ along $(1,1,1,1)$

Get the tesselation of $\mathbb{R}^3$ by squashed cubes





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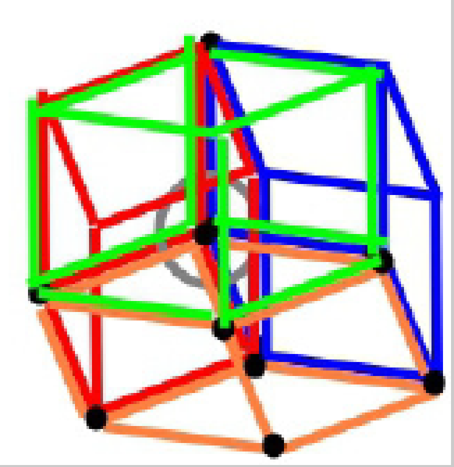
Random 3d geometries!



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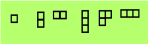
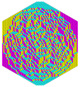
Our account of solid partitions = 4d Young diagrams

Previous famous attempts due to P. MacMahon, 1916

$Z_2(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$, L. Euler





$Z_3(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$, MacMahon

$Z_4(q) =^? \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{n(n+1)/2}}$

Gives $1, 4, 10, 26, 59, 141, \ldots, 217554, 424148 \ldots$

Instead of $\mathbf{1}, \mathbf{4}, \mathbf{10}, \mathbf{26}, \mathbf{59}, \mathbf{140}, \ldots, \mathbf{214071}, \mathbf{416849} \ldots$

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Supersymmetric count of solid partitions

Four dimensional Young diagrams

as instanton configurations in

super-Yang-Mills theory on $\mathbb{R}^8 \widetilde{\times} S^1$

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Magnificent Four Partition Function

$$Z_{8+1}^{U(n|n)}(\underline{q}; \underline{\mathbf{a}} \,|\, \underline{\mathbf{b}}; p) \; =$$

$$= \sum_{k=0}^{\infty} p^k \, \mathrm{Tr}_{\mathcal{H}_k} \left((-1)^F \; \mathbf{b}^{R_\Upsilon} \mathbf{a}^{R_I} \; \prod_{\alpha=1}^{4} q_\alpha^{R_{2\alpha-1,2\alpha}} \right)$$

$$\underline{q} = \mathrm{diag}\,(q_1, q_2, q_3, q_4) \;, \; \prod_{a=1}^{4} q_a = 1 \,,$$

parameters of an $SU(4) \subset Spin(8)$ rotation of $\mathbb{R}^8$

$$\underline{\mathbf{b}} \in U(1)^n \subset U(M) \,, \; \underline{\mathbf{a}} \in U(1)^n \subset U(N),$$

parameters of constant gauge transformations/separations of D8-branes

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Magnificent Four Partition Function

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$$\underline{\mathbf{b}} \in U(1)^n \subset U(M) \, , \ \underline{\mathbf{a}} \in U(1)^n \subset U(N),$$

parameters of constant gauge transformations/separations of D8-branes
$=$ eigenvalues of complexified $U(n|n)$ holonomy on S^1

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Magnificent Four Partition Function

from fixed points

$$Z^{U(n|n)}_{8+1}(\underline{q}; \underline{\mathbf{a}} \,|\, \underline{\mathbf{b}}; p) =$$

$$= \sum_{k=0}^{\infty} \left(\frac{p}{\sqrt{\mu}} \right)^k \sum_{\rho\,, |\rho|=k} \mathrm{Res}_{x_I = \mathrm{lexordered\ q-contents\ of}\ \rho}\ \mathbf{m}_\rho(x)$$

$$\mu = \prod_{a=1}^{n} \frac{\mathbf{b}_a}{\mathbf{a}_a}$$

measure $\mu_\rho(x)$

$$\mathbf{m}_\rho(x) = \prod_{1 \leq I, J \leq k}^{\prime} E_q(x_I / x_J) \prod_{I=1}^{k} \prod_{a=1}^{n} \frac{x_I - \mathbf{b}_a}{x_I - \mathbf{a}_a}$$

$$E_q(x) = \frac{q_4 (x - q_1 q_2)(x - q_1 q_3)(x - q_2 q_3)}{(x - q_1)(x - q_2)(x - q_3)(x - q_4)} ,$$

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Magnificent Four Partition Function

from fixed points

$$Z_{8+1}^{U(n|n)}(\underline{q};\underline{\mathbf{a}}\,|\,\underline{\mathbf{b}};p) =$$

$$= \sum_{k=0}^{\infty}\left(\frac{p}{\sqrt{\mu}}\right)^k \sum_{\rho\,,|\rho|=k} \mathrm{Res}_{x_I=\text{lexordered q}-\text{contents of }\rho}\ \mathbf{m}_\rho(x)$$

one-loop induced measure $\mathbf{m}_\rho(x)$

$$\mathbf{m}_\rho(x) = \prod_{1\le I,J\le k}^{\prime} E_q(x_I/x_J) \prod_{I=1}^{k}\prod_{a=1}^{n} \frac{x_I - \mathbf{b}_a \swarrow \text{ contribution of } \Upsilon_a}{x_I - \mathbf{a}_a \nwarrow \text{ contribution of } I_a}$$

$$E_q(x) = \frac{q_4(x-q_1q_2)(x-q_1q_3)(x-q_2q_3) \swarrow \text{ contributions of equations}}{(x-q_1)(x-q_2)(x-q_3)(x-q_4) \nwarrow \text{ contributions of B}-\text{matrices}},$$

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Magnificent Four Partition Function

Conjecture

$$Z_{8+1}^{U(n|n)}(\underline{q};\underline{\mathbf{a}}\,|\,\underline{\mathbf{b}};p) \;=\; \exp\sum_{k=1}^{\infty}\frac{1}{k}F_9(\underline{q}^k,\mu^k,p^k)$$

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Magnificent Four Partition Function

$$Z_{8+1}^{U(n|n)}(\underline{q}; \mathbf{\underline{a}} \,|\, \mathbf{\underline{b}}; p) \;=\; \exp \sum_{k=1}^{\infty} \frac{1}{k} F_9(\underline{q}^k, \mu^k, p^k)$$

Free energy $F_9(\underline{q}, \mu, p) = \dfrac{[q_{12}][q_{13}][q_{23}][\mu]}{[q_1][q_2][q_3][q_4][\sqrt{\mu p}][\sqrt{\mu / p}]}$

$$[\xi] := \xi^{\frac{1}{2}} - \xi^{-\frac{1}{2}}$$

Our formula has been checked for up to $n = 16$ instantons

with N. Piazzalunga

R. Poghossian $--$*up to* n=17

Works in all **1, 4, 10, 26, 59, 140, . . . , 214071, . . .** cases!

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Magnificent Four Partition Function

For special values of μ our partition function

Reduces to the previously known lower dimensional ones

In particular, for if $\mathbf{b}_a = q_4 \mathbf{a}_a$ for all $a = 1, \ldots, n$

we get the partition function of $U(n)$ theory in $6+1$ dimensions

which matches sugra on $\mathbb{R}^4/\mathbb{Z}_n \times \mathbb{R}^6 \times S^1$

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Magnificent Four Partition Function

For general values of μ our partition function

Coincides with that of some system of free bosons and fermions

Which contains (cohomologically) 11d linearized supegravity

What is its minimal number of spacetime dimensions?

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Beyond eleven dimensions ?!?!

Non-Poincare supersymmetry?

Beyond eleven dimensions ?!?!

Non-Poincare supersymmetry in 12+1 dimensions?

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THANK YOU

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