From 3d dualities to 2d free field correlators and back and an E-string surprise

Sara Pasquetti

Milano-Bicocca University

String-Math 2019, Uppsala

GA: 637844

based on 1712.08140 with Nedelin and Zenkevich,
1812.08142 with Aprile and Zenkevich
and 1903.10817, 1905.05807 with Sacchi
and 1907.XXXXX with Razamat, Sacchi and Zafrir

This talk deals with IR dualities and correspondences.

- IR dualities, like Seiberg duality, relate theories with different UV descriptions and same IR behavior.
  Need non-perturbative test: anomalies, operator maps, map exact quantities calculated via localisation.

- Correspondences, like AGT or gauge-BPS, establish a dictionary to map exact quantities in supersymmetric gauge theories to objects in different systems such as 2d CFT, TQFT or integral systems.

Localization

Imported from maths to physics by Witten in the '80s, had many amazing applications to supersymmetric QFTs,

for example the calculation of instanton partition functions completed by Nekrasov in 2003

and more recently the application to theories on compact manifolds initiated by Pestun in 2007.

Localising a SUSY theory on a curved background is a two step process:

- I) Formulate the theory on the curved background preserving some SUSY.

  First works did so by adding extra $1/r$ terms to the action,

  a systematic approach has been initiated by Festuccia and Seiberg.

- II) Calculate one-loop determinants $\rightarrow$ special functions

The abundance of results obtained in the localisation revival 2007-2017 and their many applications have been partially collected in a review.

Localization techniques in quantum field theories

Vasily Pestun[1], Maxim Zabzine[2], Francesco Benini[3,4], Tudor Dimofte[5], Thomas T. Dumitrescu[6], Kazuo Hosomichi[7], Seok Kim[8], Kimyeong Lee[9], Bruno Le Floch[10], Marcos Mariño[11], Joseph A. Minahan[2], David R. Morrison[12], Sara Pasquetti[13], Jian Qiu[2,14,15], Leonardo Rastelli[16], Shlomo S. Razamat[17], Silvu S. Pufu[18], Yuji Tachikawa[19], Brian Willett[20], Konstantin Zarembo[21,2]

[1] *Institut des Hautes Études Scientifique, France*
[2] *Department of Physics and Astronomy, Uppsala University, Sweden*
[3] *International School for Advanced Studies (SISSA), via Bonomea 265, 34136 Trieste, Italy*
[4] *Blackett Laboratory, Imperial College London, London SW7 2AZ, United Kingdom*
[5] *Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo, ON N2J 2Y5, Canada (on leave) Department of Mathematics, University of California, Davis, CA 95616, USA*
[6] *Department of Physics, Harvard University, Cambridge, MA 02138, USA*
[7] *Department of Physics, National Taiwan University, Taipei 10617, Taiwan*
[8] *Department of Physics and Astronomy & Center for Theoretical Physics, Seoul National University, Seoul 151-747, Korea*
[9] *School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea*
[10] *Princeton Center for Theoretical Science, Princeton University, Princeton, NJ 08544, USA*
[11] *Département de Physique Théorique et Section de Mathématiques Université de Genève, Genève, CH-1211 Switzerland*
[12] *Departments of Mathematics and Physics, University of California, Santa Barbara Santa Barbara, CA 93106 USA*
[13] *Dipartimento di Fisica, Università di Milano-Bicocca, Piazza della Scienza 3, I-20126 Milano, Italy*
[14] *Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany*
[15] *Department of Mathematics, Uppsala University, Box 480, SE-75106 Uppsala, Sweden*
[16] *C. N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, NY 11794-3840, USA*
[17] *Department of Physics, Technion, Haifa, 32000, Israel*
[18] *Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA*
[19] *Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo, Kashiwa, Chiba 277-8583, Japan*
[20] *KITP/UC Santa Barbara*
[21] *Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden*

The special functions feast

A surprising harmony emerges from the calculation of determinants.

For example for spheres with *squashing* parameters $\epsilon_1 \cdots \epsilon_r$ the localised partition functions are written in terms of two types of special functions:

$$\Upsilon_r(x|\epsilon) = \gamma_r\,(x|\epsilon)\,\gamma_r(\sum_i \epsilon_i - x|\epsilon)^{(-1)^r}\,, \quad S_r(x|\epsilon) = \gamma_r(x|\epsilon)\gamma_r(\sum_i \epsilon_i - x|\epsilon)^{(-1)^{r-1}}\,.$$

with

$$\gamma_r(x|\epsilon) = \prod_{n_1 \cdots n_r = 0}^{\infty} (x + n_1\epsilon_1 + \cdots + n_r\epsilon_r)\,.$$

For even and odd spheres a vector multiplet contributes as:

$$Z_{S^{2r}} = \int \prod_{i=1}^{N} da_i \prod_{i<j} \Upsilon_r(a_i - a_j|\epsilon)\; e^{P_r(a)} + \cdots\;,$$

$$Z_{S^{2r-1}} = \int \prod_{i=1}^{N} da_i \prod_{i<j} S_r(a_i - a_j|\epsilon)\; e^{P_r(a)} + \cdots\;.$$

Some of these integrals have been studied by mathematicians like Spiridonov and Rains $\rightarrow$ identities for Seiberg-like dualities!

I made the point that IR dualities and correspondences are two very different things, but I will now try to show that actually they are somehow related.

3d$\to$ 2d reduction of $\mathcal{N}=2$ theories

Quite subtle! problems with non-compact branches, role of the metric...[Aharony-Razamat-Seiberg-Willett],[Aharony-Razamat-Willett].

We focus on $D^2 \times S^1$ or $S^2 \times S^1$ partition functions of mass deformed theories, written in terms of $(x;q)_\infty = \prod_{k=1}^{\infty}(1-xq^k)$, where $q \sim e^R$.

2d limits depend on how the 3d real masses scale with R:

- Higgs or natural limit: matter fields remain light, we get a 2d gauge theory. At the level of partition functions

$$\lim_{q\to 1} \frac{(q^x;q)_\infty}{(q^y;q)_\infty} = (1-q)^{y-x}\,\frac{\Gamma(y)}{\Gamma(x)}\,,$$

  where S^2 and D^2 partition functions are written in terms of $\Gamma(x)$.
- Coulomb un-natural limit: matter fields are heavy, we get Landau-Ginsburg models of 2d twisted multiplets. At the level of partition functions

$$\lim_{q\to 1} \frac{(q^a x;q)_\infty}{(q^b x;q)_\infty} = (1-x)^{b-a}\,.$$

  However we can map these integrals to 2d free field correlators!

Reducing dual pairs

Reducing partition functions of dual theories, we find the following scenario:

- Higgs limit of a 3d Seiberg-like duality identity yields a similar 2d Seiberg-like duality identity.
- Coulomb limit of a 3d Seiberg-like duality identity yields a duality identity between LG models or a duality identity between free field correlators.
- For a mirror or a spectral dual pair, Higgs limits on one side becomes Coulomb limit on the other side and we get a Hori-Vafa-like duality identity or a gauge/free field correspondence.

WARNING! The duality identities above do not necessarily represent 2d dualities! [Aharony-Razamat-Willett]

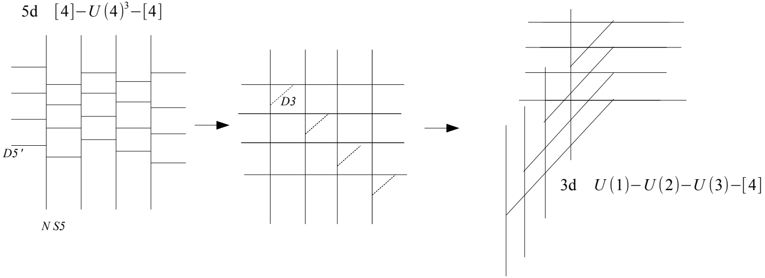
I will focus on the relation between mass deformed theories and free field correlators, 3d or 2d mass parameters are mapped to momenta and insertion points.

Let's start with how we recover gauge/free field correlators from 3d spectral dualities.

3d defect theories and dualities via Higgsing

A $U(N)^{N-1}$ 5d quiver is realised on a web of N NS5 and N D5' branes.

When the parameters are tuned to special values $\equiv$ Higgsing condition the NS5 branes can be removed from the web and D3 can be stretched.



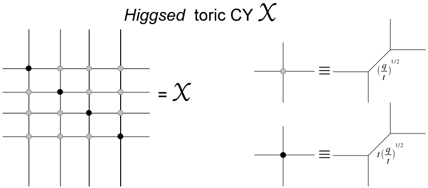
The 3d theory on the D3s is our defect theory $FT[SU(N)]$, which is $T[SU(N)]$ with an extra set of singlets and $\delta\mathcal{W} = Q_i\tilde{Q}_jX_{ij}$.

The 5d theory is invariant under S-duality (NS5 $\leftrightarrow$ D5') $\Rightarrow$ $FT[SU(N)]$ is invariant under $3d$ spectral duality.

Higgsing and geometric transition

The 5d quiver theory can be geometrically engineered in M theory on $X \times \mathbb{R}^4_{q,t} \times S^1$ where X is a toric Calabi-Yau.

The Higgsing conditions translate into quantisation conditions for the Kählers parameters of X. At these values there is geometric transition:



Using the refined topological vertex [Iqbal-Kozcaz-Vafa] we can calculate the partition function of the *Higgsed* CY $\mathcal{X}$ and compare with the localization result:

$$Z^{\mathcal{X}}_{\rm top}(\vec{\mu}, \vec{\tau}; q, t) = \mathcal{B}^{D_2 \times S^1}_{FT[SU(N)]}(\vec{\mu}, \vec{\tau}, t; q)$$

$\vec{\mu}, \vec{\tau}$ are identified with fiber and base Kähler parameters.

So we have two totally different tools to compute 3d partition functions.

3d duality from fiber-base duality

The CYs X, $\mathcal{X}$ (before and after Higgsing) are invariant under the action of fiber-base or 5d S-duality which swaps μ_i with τ_i and so

$$
\begin{array}{ccc}
Z_{top}^{\mathcal{X}}(\vec{\mu},\vec{\tau};q,t) & \xleftrightarrow[\text{DUAL}]{\text{FIBER-BASE}} & Z_{top}^{\mathcal{X}'}(\vec{\tau},\vec{\mu};q,t) \\
\| & & \| \\
B_{FT[SU(N)]}^{D^2\times S^1}(\vec{\mu},\vec{\tau},t;q) & \xleftrightarrow[\text{DUAL}]{\text{SPECTRAL}} & B_{FT[SU(N)]}^{D^2\times S^1}(\vec{\tau},\vec{\mu},t;q)
\end{array}
$$

$\rightarrow$ 3d self-duality for $FT[SU(N)]$ descends from fiber-base duality.

Reducing a spectral dual pair on $D^2 \times S^1$

- Higgs limit reduces $FT[SU(N)]$ theory to the same theory in 2d:

$$\lim_{q\to 1} \mathcal{B}^{D_2\times S^1}_{FT[SU(N)]}(\vec{\tau},\vec{\mu},t) = \mathcal{B}^{D_2}_{FT[SU(N)]}(\vec{\tau},\vec{f},\beta)\,.$$

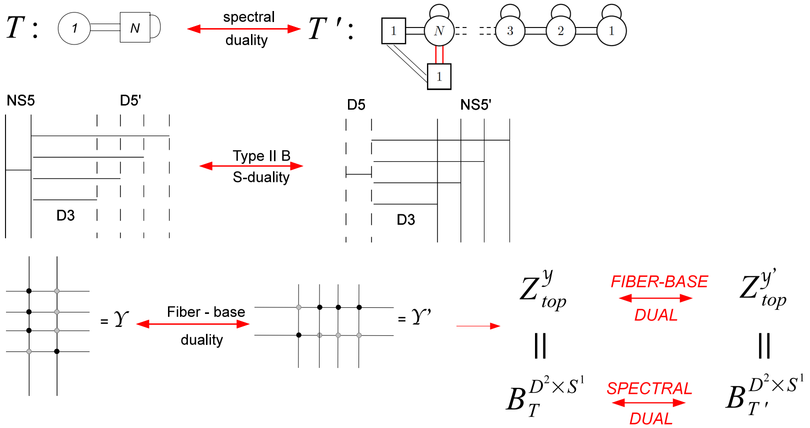
- On the dual side the Higgs limit becomes a Coulomb limit which yields an $N+2$ point free field correlator:

$$\lim_{q\to 1} \check{\mathcal{B}}^{D_2\times S^1}_{FT[SU(N)]}(\vec{\tau},\vec{\mu},t) = \langle\vec{\alpha}_\infty|\, V_{\vec{\alpha}_1}(z_1)\dots V_{\vec{\alpha}_N}(z_N)\, \left(\prod_{a=1}^{N} Q^a_{(a)}\right) |\vec{\alpha}^{(0)}\rangle^{A_{N-1}}_{\mathrm{free}}\,.$$

(screening charges$\leftrightarrow$ranks, insertion points$\leftrightarrow$masses, ...)

$\rightarrow$ Reducing the 3d spectral dual pair yields Gauge/free field correspondence. We recover the AGT map between the 2d GLSM describing the defect theory and the Toda correlator with degenerate operators [Gomis-Le Floch].

More spectral dual pairs



We can generate many new 3d spectral dual pairs from fiber-base via Higgsing. [Aprile-SP-Zenkevich]

Upon 2d reduction they originate gauge/free field correspondences.

Not too surprising given that $D^2 \times S^1$ partition functions can be mapped to q-deformed free field correlators [Aganagic-Haouzi-Kozcaz-Shakirov].

Let now see how starting from 3d Seiberg-like dualities we obtain duality identities between 2d free field correlators and what we can learn from this observation.

3d IR Seiberg-like dualities

- Aharony duality:

  $\mathcal{T}$: $U(N_c)$ with N_f flav. $Q, \tilde{Q}$, $\mathcal{W} = 0$

  $\mathcal{T}'$: $U(N_f - N_c)$ with N_f flav. $q, \tilde{q}$, $\mathcal{W} = S_- \hat{\mathfrak{M}}^+ + S_+ \hat{\mathfrak{M}}^- + Mq\tilde{q}$

- Monopole duality I: [Benini-Benvenuti-SP]

  $\mathcal{T}_{\mathfrak{M}}$: $U(N_c)$ with N_f flav. $Q, \tilde{Q}$, $\mathcal{W} = \mathfrak{M}^+$

  $\mathcal{T}'_{\mathfrak{M}}$: $U(N_f - N_c - 1)$ with N_f flav. $q, \tilde{q}$, $\mathcal{W} = \hat{\mathfrak{M}}^- + S_+ \hat{\mathfrak{M}}^+ + Mq\tilde{q}$.

- Monopole duality II: [Benini-Benvenuti-SP]

  $\mathcal{T}_{\mathfrak{M}}$: $U(N_c)$ with N_f flav. $Q, \tilde{Q}$, $\mathcal{W} = \mathfrak{M}^+ + \mathfrak{M}^-$

  $\mathcal{T}'_{\mathfrak{M}}$: $U(N_f - N_c - 2)$ with N_f flav. $q, \tilde{q}$, $\mathcal{W} = \hat{\mathfrak{M}}^+ + \hat{\mathfrak{M}}^- + Mq\tilde{q}$

Can all be obtained from 4d Intriligator-Pouliot duality for $Usp(2N)$ theories with $2N_f$ flavors via 4d$\rightarrow$3d reduction + real mass deformations.

From 3d dualities to DF duality relations

For example, by carefully taking the Coulomb limit of the Monopole duality II on $S^2 \times S^1$ we land on an identity between complex integrals found by [Fateev-Litvinov]:

$$\int \prod_{i=1}^{N_c} d^2 z_i \prod_{i<j}^{N_c} |z_i - z_j|^2 \prod_{i=1}^{N_c} \prod_{a=1}^{N_f} |z_i - t_a|^{2p_a} = \prod_{a=1}^{N_f} \gamma(1+p_a) \prod_{a<b}^{N_f} |t_a - t_b|^{2(1+p_a+p_b)} \times$$

$$\times \int \prod_{i=1}^{N_f-N_c-2} d^2 z_i \prod_{i<j}^{N_f-N_c-2} |z_i - z_j|^2 \prod_{i=1}^{N_f-N_c-2} \prod_{a=1}^{N_f} |z_i - t_a|^{-2(1+p_a)} ,$$

with the condition $\sum_{a=1}^{N_f} p_a = -N_c - 1$ and

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} ,$$

This identity and its various specialization are the so called duality relations for the Dotesenko-Fateev integrals appearing in various CFT context.

Liouville correlators from free fields

[Goulian-Li] observed that in Liovulle CFT (a 2d boson with $e^{b\phi}$ interaction) correlator exhibit poles. For example:

$$\underset{\alpha_1+\alpha_2+\alpha_3=Q-Nb}{\text{res}}\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle = (-\pi\mu)^N I_N(\alpha_1,\alpha_2,\alpha_3)$$

at these values the correlator is represented by Dotsenko-Fateev free field correlator with N screening charges:

$$I_N(\alpha_1,\alpha_2,\alpha_3) = \langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)(Q)^N\rangle_{free}\,, \qquad Q = \int dx\ e^{b\phi}$$

expanding the free field in oscillators and normal-ordering we are left with the evaluation of a complex integral:

$$I_N(\alpha_1,\alpha_2,\alpha_3) = \int \prod_{k=1}^{N} d^2\vec{t}_k\ |t_k|^{-4b\alpha_1}|t_k-1|^{-4b\alpha_2}\prod_{i<j}^{N}|t_i-t_j|^{-4b}\,.$$

To determine $\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle$ for generic momenta lifting the screening constraint we need to be able to do analytic continuation in N.

To evaluate the integral one can use the *duality relations* between complex DF integrals and find a recursion [Fateev-Litvinov]

$$I_N(\alpha_1,\alpha_2,\alpha_3) = \frac{\gamma(-Nb^2)}{\gamma(-b^2)} \frac{1}{\gamma(2b\alpha_1)\gamma(2b\alpha_2)\gamma(2b\alpha_3+(N-1)b^2)}\ I_{N-1}(\alpha_1+b/2,\alpha_2+b/2,\alpha_3).$$

Repeating this procedure N times, we obtain the evaluation formula:

$$I_N(\alpha_1,\alpha_2,\alpha_3) = \prod_{k=1}^{N}\left(\frac{\gamma(-kb^2)}{\gamma(-b^2)}\right)\prod_{j=0}^{N-1}\frac{1}{\gamma(2b\alpha_1+jb^2)\gamma(2b\alpha_2+jb^2)\gamma(2b\alpha_3+jb^2)}.$$

We can rewrite I_N in a form parametric in N which allows for analytic continuation, lifting the screening condition:

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle = C(\alpha_1,\alpha_2,\alpha_3) \sim \frac{\Upsilon'(0)\prod_{k=1}^{3}\Upsilon(2\alpha_k)}{\Upsilon(\alpha-Q)\prod_{k=1}^{3}\Upsilon(\alpha-2\alpha_k)},$$

where $\Upsilon(x)$ satisfies the functional relations

$$\Upsilon(x+b) = \gamma(bx)b^{1-2bx}\Upsilon(x) \qquad \text{same for } b\to b^{-1}\,.$$

This is the celebrated Dorn-Otto-Zamolodchikov-Zamolodchikov formula.

3d dualities *from* 2d DF duality relations?

We saw that 3d Seiberg-like dualities reduce to the DF duality relations appearing in the derivation of the Liouville 3-point function.

Can we reverse the logic? Can we *uplift* to a genuine 3d duality also the evaluation formula for the I_N integral?

Yes!

The *uplift* we are looking is a recently proposed 3d $\mathcal{N}=2$ duality [Benvenuti]:

- **Theory A**: $U(N)$ with adjoint Φ, one flavor P, $\tilde{P}$, N singlets β_j:

$$\mathcal{W}=\sum_{j=1}^{N}\beta_j \mathrm{Tr}\Phi^j\,.$$

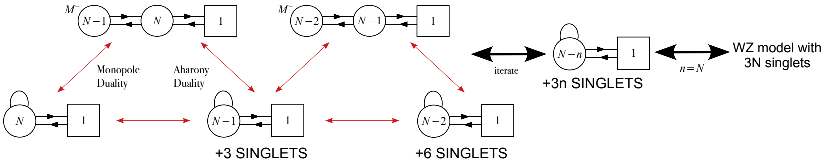
- **Theory B**: Wess-Zumino model with $3N$ singlets α_j, $T_j^{\pm}$:

$$\hat{\mathcal{W}}=\sum_{i,j,l=1}^{N}\alpha_i T_j^+ T_{N-l+1}^- \delta_{i+j+l,2N+1}\,.$$

Global symmetry: $U(1)_\tau \times U(1)_\zeta \times U(1)_\mu$. Operator map:

$$\begin{array}{rcll}
\textbf{Theory A} & \leftrightarrow & \textbf{Theory B} & \\
\mathfrak{M}^+_{\Phi^s} & \leftrightarrow & T^+_{s+1} & \\
\mathfrak{M}^-_{\Phi^s} & \leftrightarrow & T^-_{N-s} & \\
\mathrm{Tr}\left(\tilde{P}\Phi^s P\right) & \leftrightarrow & \alpha_{s+1}, & \qquad s=0,\cdots,N-1\,.
\end{array}$$

We can prove this duality by iterating 3d basic dualities:



Localised partition functions on any 3-manifold match. For example:

$$\mathcal{Z}^{S^3}_{U(N)} = \int \prod_{a=1}^{N} dx_a e^{2\pi i\zeta \sum_a x_a} \frac{\prod_{a,b=1}^{N} S_2\left(Q - i(x_a - x_b) + 2i\tau\right)}{\prod_{a<b}^{N} S_2\left(Q \pm i(x_a - x_b)\right)} S_2\left(Q \pm ix_a + i\mu\right) =$$

$$= \prod_{j=1}^{N} S_2\left(Q + 2ij\tau\right) S_2\left(Q + 2i\mu + 2i(j-1)\tau\right) \times$$

$$\times S_2\left(\frac{Q}{2} + i\zeta - i\mu - 2i(N-j)\tau\right) S_2\left(\frac{Q}{2} - i\zeta - i\mu - 2i(j-1)\tau\right) = \mathcal{Z}^{S^3}_{WZ}\,,$$

where $S_2(x|\omega_1,\omega_2) \equiv S_2(x)$ is the function appearing in 1-loop computations on a 3-sphere $\omega_1^2|z_1|^2 + \omega_2^2|z_2|^2 = 1$.

So we succeeded in *uplifting* the evaluation formula for the 3-point correlator to a genuine 3d IR duality.

What about the analytic continuation? can we make sense of it?

YES!

The function $S_3(z|\omega_1,\omega_2,\omega_3)$ appears in 1-loop computations on a 5-sphere $\omega_1^2|z_1|^2+\omega_2^2|z_2|^2+\omega_3^2|z_3|^2=1$ and satisfies:

$$S_3(z+\omega_3|\omega_1,\omega_2,\omega_3)=S_2(z|\omega_1,\omega_2)S_3(z|\omega_1,\omega_2,\omega_3)$$

which allows us to rewrite the 3d WZ partition function as

$$\mathcal{Z}_{WZ}^{S^3}=\operatorname*{Res}_{N\in\mathbb{N}}\frac{S_3'\left(0\right)S_3\left(-2i\mu+2i\tau\right)S_3\left(\frac{Q}{2}\pm i\zeta-i\mu-2i(N-1)\tau\right)}{S_3\left(-2iN\tau\right)S_3\left(-2i\mu\right)S_3\left(\frac{Q}{2}\pm i\zeta-i\mu+2i\tau\right)}=\operatorname*{Res}_{N\in\mathbb{N}}\mathcal{Z}_{T_2}^{S^5}\,.$$

On the RHS we recognise the partition function of the 5d T_2 theory!

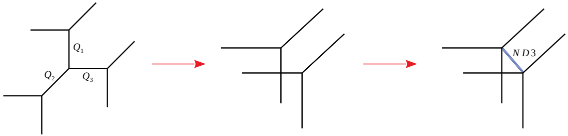
We identify $\omega_3=2i\tau$ while μ, ζ and $N\tau$ with the fugacities for the global symmetry $SU(2)^3$ of T_2.

How do we interpret the quantization condition of the fugacity in the 5d theory?

Analytic continuation as geometric transition

The $5d$ T_N theory can be realised on a web of intersecting $(0,1)$-branes, $(1,0)$-branes and $(1,1)$-branes [Benini-Benvenuti-Tachikawa]:

Equivalently we can geometrically engineer T_N by M-theory on the toric Calabi-Yau tree-fold $\frac{\mathbb{C}^3}{\mathbb{Z}_N \times \mathbb{Z}_N}$ with toric diagram given by the brane web.



The quantization condition on the parameters implies that one Kähler parameter is quantized $Q = q^N q^{1/2} t^{-1/2}$. This leads to geometric transition.

We arrive at a configuration of N D3 branes stretched between two pieces of the web. The defect theory living on the N D3 branes is our $3d$ $U(N)$ theory with one adjoint and one flavor.

As expected from the triality of [Aganagic-Haouzi-Shakirov].

This curious connection between the 3-point Liouville correlator and the 3d $U(N)$ theory is just the starting point.

We can get many new 3d dualities *from* free field correlators.

For example the $k+3$-point free field correlator [Fateev-Litvinov].

$$\underset{\alpha_1+\alpha_2+\alpha_3=Q-\frac{k}{2}-Nb}{\mathrm{res}} \langle V_{-\frac{b}{2}}(z_1)\cdots V_{-\frac{b}{2}}(z_k) V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty)\rangle \quad = \\ = (-\pi\mu)^N I_N^k(\alpha_1,\alpha_2,\alpha_3)$$

the rank N integral can be massaged in a form suitable for analytic continuation using the Kernel function:

$$I_N^k(\alpha_1,\alpha_2,\alpha_3) \sim \int \prod_{i=1}^{k} d^2\vec{x}_i \prod_{i<j}^{k} |x_i-x_j|^{-4b}|x_i|^{2A}|x_i-1|^{2B} K_c^k(x_1,..,x_k|z_1,..,z_k)\,,$$

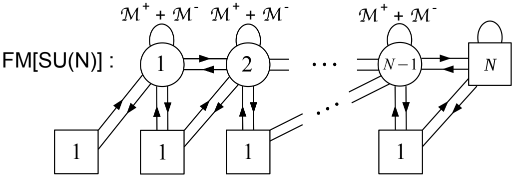
On the rhs N enters only parametrically inside A, B, C allowing for analytic continuation.

The kernel function satisfies remarkable properties. For example:

$$K_c^k(x_1,..,x_k|z_1,..,z_k) = K_c^k(z_1,..,z_k|x_1,..,x_k)$$

We can *uplift* all this to 3d QFT.

$FM[SU(N)]$: the 3d $\mathcal{N}=2$ avatar of the kernel function

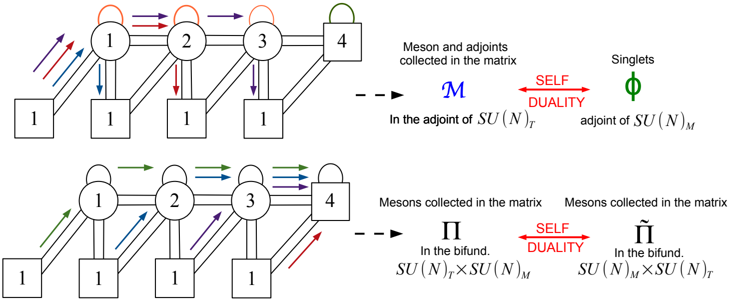


- Global IR symmetry group: $SU(N)_M \times SU(N)_T \times U(1)_A \times U(1)_\Delta$
- Reduces to $FT[SU(N)]$ when real mass for $U(1)_\Delta$ is turned on
- The $S^2 \times S^1$ partition function of $FM[SU(N)]$ reduces in the 2d Coulomb limit to $K_N^C(x_1, .., x_N | z_1, .., z_N)$

Self-duality

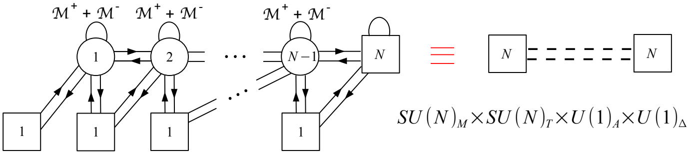
$FM[SU(N)]$ is invariant under a duality swapping $SU(N)_T \leftrightarrow SU(N)_M$

- Operator map



- Explicit map of partition functions (iterating fundamental identities) up tp $N = 2$. Match various orders of the R-symmetry fuguacity expansion of the $S^2 \times S^1$ index.

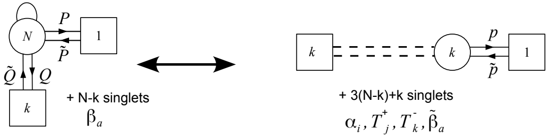
can we construct interesting $3d$ $\mathcal{N} = 2$ theories using $FM[SU(N)]$ as a building block?



Note we can gauge manifest and emergent symmetries.

3d duality *from* the 3+k-point correlator identity

GLOBAL SYMMETRY GROUP $U(k)\times U(1)^3$



$$
\begin{array}{lcl}
\mathrm{Tr}_N \Phi^{N-k+a} & \leftrightarrow & \tilde{\beta}_a, \qquad a = 1, \cdots, k \\
\mathfrak{M}^+_{\Phi^s} & \leftrightarrow & \begin{cases} T^+_{s+1} & s = 0, \cdots, N-k-1 \\ \mathfrak{M}^+_{\mathbb{M}^{k-N+s}} & s = N-k, \cdots, N \end{cases} \\
\mathfrak{M}^-_{\Phi^s} & \leftrightarrow & \begin{cases} T^-_{N-s} & s = 0, \cdots, N-k-1 \\ \mathfrak{M}^-_{\mathbb{M}^{k-N+s}} & s = N-k, \cdots, N \end{cases} \\
\mathrm{Tr}_N \left(\tilde{P}\Phi^s P\right) & \leftrightarrow & \begin{cases} \alpha_{s+1} & s = 0, \cdots, N-k-1 \\ \mathrm{Tr}_k \left(\tilde{p}\mathbb{M}^{k-N+s} p\right) & s = N-k, \cdots, N-1 \end{cases}
\end{array}
$$

$$
Q\tilde{P} \leftrightarrow \tilde{\Pi}\tilde{p}\,, \qquad P\tilde{Q} \leftrightarrow p\Pi\,, \qquad Q\tilde{Q} \leftrightarrow \mathcal{M}\,.
$$

Tests: partition functions match (iterating fundamental identities) up to $k = 2$ and perturbative match of the index up to $k = 3$.

More?

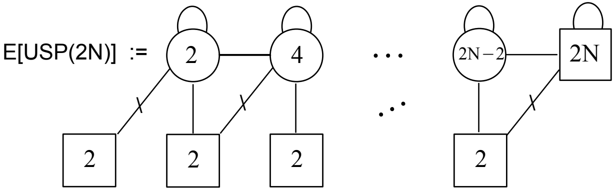
Using the $FM[SU(N)]$ we can construct various interesting 3d theories with several dual frames.

We can also remember that 3d basic dualities, Aharony, Monopole I and II can be obtained from 4d$\rightarrow$3d compactification + real mass deformation from 4d Seiberg-like dualities.

It is then natural to try to uplift to 4d. In particular, is there a 4d ancestor of the kernel?

$E[Usp(2N)]$: the 4d $\mathcal{N}=1$ avatar of the kernel function

4d $\mathcal{N}=1$ quiver with $Usp(2n)$ nodes:

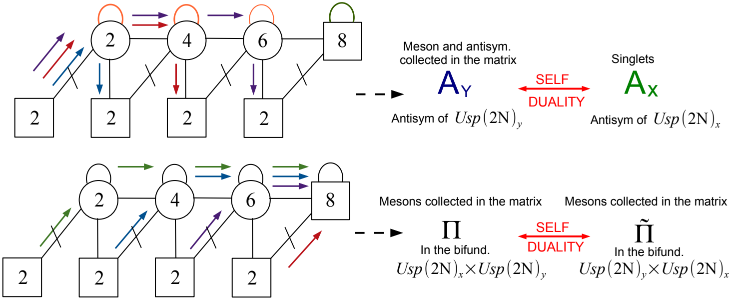


- Global IR symmetry $USP(2N)_x \times USP(2N)_y \times U(1)_t \times U(1)_c$
- Upon circle compactification+ real mass flows to $FM[SU(N)]$
- The index coincides with the Rains' interpolation kernel: $\mathcal{I}_{E[Usp(2N)]}(x,y,c,t) = \mathcal{K}_c(x,y,t) \sim \sum_\lambda \Delta_\lambda(c) R^*_\lambda(x) R^*_\lambda(y)$

Self-duality

$E[Usp(2N)]$ is invariant under a duality swapping $Usp(2N)_x \leftrightarrow Usp(2N)_y$

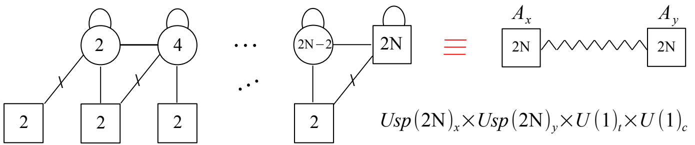
- Operator map:



- For the index from one of the Rains' kernel identities we have:

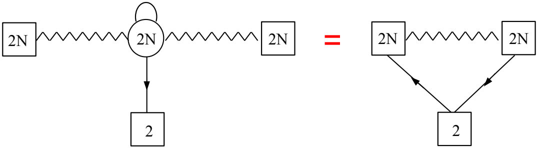
$$\mathcal{I}_{E[Usp(2N)]}(x, y, c, t) = \mathcal{I}_{E[Usp(2N)]}(y, x, c, t)$$

The $E[Usp(2N)]$ building block



Special cases:

Braid:



Reduces to Seiberg duality for $N = 1$.

Can we construct interesting 4d $\mathcal{N} = 1$ theories using the $E[Usp(2N)]$ building block?

YES, the torus compactifications of the rank-N E-string theory!

$6 = 4 + 2$ program

Realizing 4d SCFT from 6d theories, can be very insightful: importance of non-Lagrangian theories, symmetry enhancements, dualities, etc...

Given a 6d theory and a choice of compactification, we can make predictions for global symmetries and anomalies for large classes of 4d theories. Can we identify them directly in 4d?

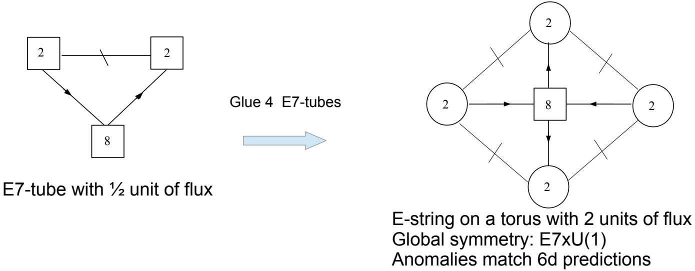
For example the 6d (1,0) Rank-N E-string theory ($E8 \times SU(2)_t$ global symmetry) can be compactified on a torus with fluxes in the $E8$ Cartan.

This setup predicts the existence of 4d $\mathcal{N} = 1$ SCFTs with global symmetry group $G_F \times SU(2)_t$, with G_F a subgroup of $E8$. It also predicts what their anomalies should be. [Kim-Razamat-Vafa-Zafrir]

(rank-one) E-string on a torus

These 4d $\mathcal{N} = 1$ theories from rank-1 E-string on a torus can be built by gluing tubes. [Kim-Razamat-Vafa-Zafrir]

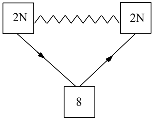
For example, for fluxes breaking $E8 \to E7 \times U(1)$:



Fluxes breaking E7 to smaller subgroups are realised by more general gluings. Higher rank E-strings?

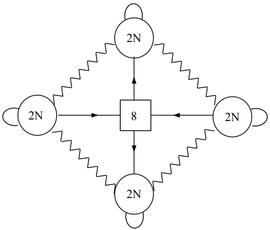
rank-N E-string on a torus

With the $E[Usp(2N)]$ theory we can construct the rank-N E-string tube:



E7-tube with ½ unit of flux

Glue 4 E7-tubes



Rank N E-string on a torus with 2 units of flux
Global symmetry: E7xU(1)xSU(2)
Anomalies match 6d predictions

Again fluxes breaking E7 to smaller subgroups are realised by more general gluings.

$\Rightarrow$ Mathematical objects like the Fateev-Litvinov kernel and the Rains's kernel can be given a new life as building blocks of susy QFTs. With them we can construct new theories with interesting symmetries and find new dualities.

$\Leftarrow$ Physics perspective where it is natural to connect different QFTs via dualities or via RG flows or compactifications could be useful to connect these mathematical objects and the mathematical worlds they come from.

THANK YOU!