Multiplicative Higgs bundles

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Higgs bundles on a curve

For a complex variety X, an effective divisor D and a complex reductive group G, a G-Higgs bundle is a pair (P, ϕ) where P is a principal G-bundle and

$$\phi \in \Gamma(X, ad(P) \otimes K_X(D))$$

Then we have

Theorem

There is an algebraically integrable system on the moduli space of meromorphic Higgs bundles on a curve [Hitchin, Markman, Bottacin, Mukai, Tyurin]

Hitchin system

A Hitchin system on a complex curve X plays a prominent role in

- the geometric Langlands program Beilinson-Drinfeld, Kapustin-Witten,
- the construction of the compactification of the 6d (2,0) self-dual theory on X viewed as 4d $\mathcal{N} = 2$ theory Gaoitto, Seiberg-Witten
- (after quantization) relation to the Nekrasov's functions of the corresponding 4d theory Nekrasov-Shatashvili
- (after hyperKahler rotation) relation to Toda theory and correlation functions determined by W-algebra Alday-Gaiotto-Tachikawa

Abstract Higgs bundles

There is a more abstract version of Hitchin system constructed in [Donagi] and [Donagi-Gatsgory].

By this construction, the usual Hitchin system on X with singularities in D is an example of a system of an abstract Higgs bundle on X valued in the line bundle $K_X(D)$.

An abstract G-Higgs bundle is a pair (P, c) where P is a principal G-bundle on X, and $c \subset \mathrm{ad}(P)$ is a subbundle. Each fiber of c consists of of centralizers in $\mathfrak{g}$ of regular elements of G.

There is an equivalence between an abstract G-Higgs bundle and an abstract spectral data $(\tilde{X}, \mathcal{T})$ consisting of an abstract cameral cover $\tilde{X}$ and a suitably transformed T-bundle on $\tilde{X}$ [Donagi-Gaitsgory]

Valued Higgs bundles

To relate to the usual definition of Higgs bundle, we consider an abstract bundle (P, c) *valued* in $K_X(D)$ by adding a section

$$\phi \in \Gamma(X, c \otimes K_X(D))$$

and to the spectral data $(\tilde{X}, \mathcal{T})$ we add W-invariant collection of maps

$$\tilde{X} \to T_X^*$$

Now, we can replace the space of *values*, the line bundle K_X, by any family Y of groups fibered over X. A Higgs field ϕ is then a section

$$\phi \in \Gamma(X, c \otimes Y)$$

To the abstract spectral data $(\tilde{X}, \mathcal{T})$ we add collection of maps $\tilde{X} \to Y$, and in this way we get a moduli space

$$Higgs_G(Y/X)$$

Elliptic fibrations

Suppose that Y is fibered over X, and a generic fiber is an elliptic curve. Then Donagi has shown that the moduli space $\mathrm{Bun}_G(Y)$ of G-bundles on Y is isomorphic with the moduli space of G-Higgs bundles on X with values in the fibration $Y \to X$.

If Y is itself a 2-dimensional algebraically integrable system fibered over X with elliptic fibers, the resulting $\mathrm{Bun}_G(Y) = \mathit{Higgs}_G(Y/X)$ is an algebraic integrable system

Vertical hierarchy

Consider now 2-dimensional Y and 1-dimensional X so that Y_x are 1-dimensional complex connected abelian groups.

There are three cases for Y_x:

- the elliptic curve $Y_x = E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$
- the multiplicative group $Y_x = G_m = \mathbb{C}^\times = \mathbb{C}/\mathbb{Z}$ (nodal degeneration of E)
- the additive group $Y_x = G_a = \mathbb{C}$ (cusp degeneration of E)

Rob Donagi's question

Consequently, we have three versions of valued abstract Higgs bundle on X:

- $Y_x = E$, and then $Higgs_G(Y/X) = Bun_G(Y)$
- $Y_x = G_m = \mathbb{C}^\times$, and then $Higgs_G(Y/X) =?$
- $Y_x = G_a = \mathbb{C}$, and then $Higgs_G(Y/X) = Hitchin_G(X)$

> Hitchin's system on the moduli of bundles on an arbitrary curve X, as well as Markman's meromorphic extension, involve Higgs bundles with arbitrary structure group G but with values in a line bundle, which is a group variety over X with fiber G_a. The moduli space of G-bundles on an elliptic fibration involves Higgs bundles with structure group G and with values in the elliptic fibration. Is there an interesting geometric interpretation of the remaining "trigonometric" case, where the values are taken in G_m?

Donagi, Geometry and integrability, 2003

Purpose of this talk

The purpose of this talk is to answer Donagi's question on an interesting geometrical interpretation of

$$mHiggs_G(X) = Higgs_G(Y/X)$$

for the multipliative case of $Y_x = G_m = \mathbb{C}^\times$.

Outline of the results: symplectic leaf

Theorem

$\mathrm{mHiggs}^{\mathrm{fr},0}_{G;g_\infty}(X;D)$ *is a symplectic leaf in Poisson-Lie rational loop group* $G_1(K(X))$ *and carries a holomorphic symplectic structure induced by rational Sklyanin Poisson structure*

- The dimension of symplectic leaf is $2\sum_{x\in D}(\check{\omega}_x,\rho)$
- There is an integrable system fibration

$$\mathrm{mHiggs}^{\mathrm{fr},0}_{G;g_\infty}(X;D) \xrightarrow{\chi} \mathcal{B}_G(X,D)$$

where the projection map is generated by evaluation characters of G, and the fibers are half-dimensional complex flat tori (possibly non-compact)

Outline of the results: algebraic integrable system

- For a semi-simple G and a regular semi-simple framing $g_\infty \in G$, there is a symplectic reduction

$$\mathrm{mHiggs}^{\mathrm{fr},0,red}_{G;g_\infty}(X;D) := \mathrm{mHiggs}^{\mathrm{fr},0}_{G;g_\infty}(X;D)//T$$

  where the Hamiltonian functions for the T action are generated by $\mathrm{res}_{x_\infty}\chi(g(x))$, and $\dim_{\mathbb{C}} \mathrm{mHiggs}^{\mathrm{fr},0,red}_{G;g_\infty}(X;D) = 2(\rho,\check{\omega}) - 2\mathrm{rank}G$
- The reduced space $\mathrm{mHiggs}^{\mathrm{fr},0,red}_{G;g_\infty}(X;D)$ carries a structure of the fibration of algebraic integrable system whose Lagrangian fibers are generically compact abelian varieties.
- A holomorphic complex structure on $\mathrm{mHiggs}^{\mathrm{fr},0,red}_{G;g_\infty}(X;D)$ can be lifted to a canonical semi-flat hyperKahler structure

Outline of the results: Periodic Monopoles and HyperKahler geometry

- There is an analytic isomorphism $\mathcal{D}$ Donaldson-Uhlenbeck-Yau / Kobayashi-Hitchin / Simpson / Charbonneau-Hurtubise/ Mochizuki between the locus of polystable multiplicative G-Higgs bundles and the moduli space of singular monopoles on flat Riemannian 3d space $X \times S^1$

$$\mathrm{mHiggs}^{\mathrm{fr},0,red,polystable}_{G;g_\infty}(X;D) \stackrel{\mathcal{D}^*}{\to} \mathrm{Mon}^{red}_{G_c}(X \times S^1; \tilde{D})$$

Theorem

The hyperKahler moduli space of singular monopoles on $X \times S^1$ viewed in twistor structure $\zeta = 0$ as holomorphic symplectic variety is isomorphic to a symplectic leaf in Sklyanin's Poisson-Lie loop group $G_1(K(X))$

$$\Omega_{Sklyanin\ \mathbf{r}-\mathrm{matrix}}(\mathrm{mHiggs}) = \mathcal{D}^*\Omega_{HK,\zeta=0}(\mathrm{Mon})$$

Outline of the results: twistor rotation and difference connections

In the additive case, usual Hitchin moduli space on X after twistor rotation is symplectomorphic to the moduli space of complex flat connections: a space of pairs (P, A) where P is a holomorphic G-bundle on X, and A is a holomorphic $(1, 0)$-connection 1-form.

Here is a parallel statement for multiplicative Higgs bundles

Theorem

In the limit $R \to \infty$ with $2\pi R\zeta = \epsilon$, the moduli space of singular monopoles on $X \times S^1_R$ at twistor point ζ is isomorphic as holomorphic symplectic variety to the moduli space of G ϵ-connections on X.

For an automorphism ϵ of X, an ϵ-connection on X on a principal G-bundle P is a morphism $P \to \epsilon^* P$. Trivialization of a G-group valued ϵ-connection $A(x)$ amounts to solving the difference equation

$$\psi(x + \epsilon) = A(x)\psi(x)$$

Conventions

We'll focus on horizontal rational case $X = \mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$ but the technique is equally well applicable to $X = \mathbb{C}^\times$ or X an elliptic curve

Fix $(X = \mathbb{P}^1, dx)$ where $dx \in \Gamma(X, K_X(2x_\infty))$ is a 1-form with a degree 2 pole at the infinity $x_\infty = \infty \in \mathbb{P}^1$.

Fix a divisor $D = (x_i, \check{\omega}_i)$ on X valued in a cone of dominant co-weights of the coweight lattice $\check{\Lambda}$.

Multiplicative Higgs bundle

Definition (Hurtubise-Markman'02)

A multiplicative G-Higgs bundle on X framed at $x_\infty \in X$ with singularities in a co-weight valued divisor D is a pair (P, g) where P is a principal G-bundle on X with a framing fixed at x_∞, and g is a section

$$g \in \Gamma(X, P \times_G \mathrm{Ad}_G)$$

of group valued adjoint bundle on X such that near each $(x_i, \check{\omega}_i) \in D$ there exists local holomorphic sections holomorphic $g_L(x), g_R(x)$ such that $g(x) = g_L(x)(x - x_i)^{\check{\omega}_i} g_R(x)$ where $\check{\omega}_i$ is the co-weight morphism $\check{\omega}_i : \mathbb{C}^\times \to T_G, x \mapsto x^{\check{\omega}_i}$ at $x_i \in D$; and $g(x_\infty) = g_\infty \in G$. The element $g_\infty \in G$ is called framing.

Singularities and orbits in affine grassmanian

More abstractly, in a formal neighborhood of each puncture $x_i \in D$, a restriction of multiplicative Higgs field $g(x)$ defines an element of algebraic loop group $LG = G((z))$ where $z = x - x_i$. This element is well defined up to the adjoint action of $L^+G = G[[z]]$. Also, the singularity class does not change under the left or right multiplication by $G[[z]]$. Consequently, the singularity class $G((z))$ element is a coset in the affine Grassmanian

$$G[[z]] \setminus G((z))/G[[z]] = L^+G \setminus \mathrm{Gr}_G$$

Orbits in affine Grassmanian are in canonical bijection with with dominant coweights of $\mathfrak{g}$, so fixing a degree a the puncture is fixing a dominant coweight $\check{\omega}_{x_i}$ at each $x_i \in D$.

Moduli space

Definition

A moduli space

$$\mathrm{mHiggs}^{\mathrm{fr}}_{G;g_\infty}(X;D)$$

is a moduli space of multiplicative G-Higgs bundles on X framed at x_∞ with singularities defined by D

These spaces have been considered before, sometimes under the name of moduli space of G-pairs. Earlier works inlcude

- Bottacin'95
- Arutyunov, Frolov, Medvedev '97
- Cherkis, Kapustin '00
- Braden, Chernyakov, Dolgushev, Levin, Olshanetsky, Zotov '03,'07
- Hurtubise, Markman '02
- Frenkel and Ngô '11
- Bouthier '14, '15

Calabi-Yau condition on X

While the structure of multiplicative Higgs bundle makes sense for any curve X, the moduli space carries canonical symplectic structure only in the very special situation, when curve X is Calabi-Yau with specified non-degenerate section of the canonical bundle K_X, and this is possible if X is flat curve, i.e. X is (a compactification of) $\mathbb{C}$, $\mathbb{C}^\times$ or elliptic curve.

One can think about $\mathbb{C}$ and $\mathbb{C}^\times$ case as a smooth open locus of a degenerate cuspidal or nodal elliptic curve respectively.

For the rest we focus on the horizontally rational case $X = \mathbb{C} \cup \{x_\infty\} = \mathbb{P}^1$ equipped with 1-form dx (it has a degree 2 pole at x_∞).

The case of X being an elliptic curve was studied in Hurtubise-Markman'02

Rational Poisson-Lie group

Let $K(X)$ be the the field of rational functions on X. Let $G(K(X))$ be an infinite-dimensional Lie group value in the field $K(X)$ of rational functions on X, and let $G_1(K(X))$ denotes a subgroup of elements framed at infinity $\{g \in G(K(X)) \quad | \quad g(x_\infty)\} = 1$.

Then $G_1(K(X))$ carries a structure of Poisson-Lie group where the Sklyanin Poisson structure is defined by the rational **r**-matrix with the kernel

$$\mathbf{r}(x_1, x_2) = \frac{\Omega}{x_1 - x_2}$$

where $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is a quadratic Casimir induced from a Killing form on $\mathfrak{g}$
Sklyanin,Reshetikhin, Reyman, Semenov-Tian-Shansky'83

Sklyanin's bracket

Given an algebraic function $\phi : G \to \mathbb{C}$ and a point $x \in X$ we consider evaluation functions $\phi_x : G(K(X)) \to \mathbb{C}$ that are compositions of ϕ and evaluation map at x

$$\phi_x : g \mapsto \phi(g(x))$$

The **r**-matrix defines Poisson bracket on the evaluation functions ϕ_{x_1}, ϕ_{x_2}.

The left and right gradients of the evaluation functions at $g \in G(K(X))$ on a vector field ξ on $G(K(X))$ are defined as

$$(\xi, \nabla_L(\phi_x)) = \frac{d}{dt}\phi(e^{\xi(x)t}g(x))|_{t=0}, \qquad (\xi, \nabla_R(\phi_x)) = \frac{d}{dt}\phi(g(x)e^{\xi(x)t})|_{t=0}$$

Then Poisson bracket of ϕ_{x_1} and ψ_{x_2} at $g \in G(K(X))$ is given by

$$\{\phi_{x_1}, \psi_{x_2}\}(g) = \frac{1}{x_1 - x_2}\left(\langle \nabla_L(\phi_{x_1}), \nabla_L(\psi_{x_2})\rangle - \langle \nabla_R(\phi_{x_1}), \nabla_R(\psi_{x_2})\rangle\right)$$

where $\langle , \rangle$ is a pairing on $\mathfrak{g}^*$ induced from a Killing form.

Adjoint invariant functions on G

Notice, that if ϕ_x is an adjoint invariant evaluation function on $G(K(X))$ then $\nabla_L \phi_x = \nabla_R \phi_x$, and consequently, if ϕ_{x_1} and ψ_{x_2} are adjoint invariant evaluation functions, then

$$\{\phi_{x_1}, \psi_{x_2}\} = 0$$

Consequently, the intersection of the image $\mathrm{mHiggs}_G^{\mathrm{fr}}(X, D) \to G_1[[x^{-1}]]$ with a fiber of the Chevalley map

$$\chi : G_1[[x^{-1}]] \to T[[x^{-1}]]/W$$

is isotropic for the Sklyanin Poisson structure.

mHiggs as a symplectic leaf

Now we'll demonstrate the main steps in the proof of the theorem

Theorem

The subvariety $\mathrm{mHiggs}^{\mathrm{fr},0}_{G;g_\infty}(X;D) \subset G_1(K(X))$ *is a symplectic leaf with respect to Sklyanin's poisson structure on* $G_1(K(X))$ *defined by the rational* $\mathbf{r}$*-matrix. Consequently,* $\mathrm{mHiggs}^{\mathrm{fr},0}_{G;g_\infty}(X;D) \subset G_1(K(X))$ *carries canonical holomorphic symplectic structure of the rational* $\mathbf{r}$*-matrix type*

First we'll consider the deformation complex and identify the tangent space to a multiplicative G-Higgs pair (P, g).

Left and Right deformations

For a group valued section $g \in \Gamma(X \setminus D, \mathrm{Ad}_P)$, a deformation in $T_g G$ of g can be identified with the tangent space at the unity of the group (the Lie algebra) either by the left or by the right multiplication.

In matrix notations (e.g. in a faithful representation) we can present the deformation as

$$\delta g = \xi^L g + g \xi^R, \qquad g \in G, \xi \in \mathfrak{g}$$

with the equivalence relation

$$(\xi^L, \xi^R) \sim (\xi^L, \xi^R) + (Ad_g \xi, -\xi)$$

where $Ad_g \xi := g \xi g^{-1}$.

That is to say that away from D, the pair (ξ^L, ξ^R) is a section of the quotient bundle $\mathfrak{a}_P$

$$\mathfrak{g}_P \xrightarrow{(Ad_g, -1)} \mathfrak{g}_P \oplus \mathfrak{g}_P \to \mathfrak{a}_P$$

There are usual gauge transformations, their infinitesimals are parameterized by sections of $\mathfrak{g}_P[1]$. Consequently, the deformation complex of multiplicative G-Higgs pairs on X with a framing at x_∞ is

$$(\mathfrak{g}_P[1] \xrightarrow{(1,-1)} \mathfrak{a}) \otimes \mathcal{O}(-x_\infty)$$

If $X = \mathbb{P}^1$, and we fix P to be a trivial bundle on X with framing at infinity, there are no non-trivial deformations of P, and non-trivial automorphisms.

Consequently, the deformation complex of multiplicative G-Higgs pairs is concentrated in degree 1.

Now we consider the tangent space T_gmHiggs and compute its dimension

Dimension of mHiggs moduli space

By the equivalence on the space of (ξ^L, ξ^R) deformation pairs we have

$$(\xi^L, \xi^R) \sim (\xi^L + Ad_g \xi, \xi^R - \xi) \sim (\xi^L + Ad_g \xi^R, 0)$$

Assume that $g(x)$ is generic near each singularity x_i, i.e. assume that $g(x)$ is regular semi-simple as $x \to x_i$. In this case the map $Ad_{g(x)}$ is diagonizable near x_i. Let x-dependent Cartan subalgebra $\mathfrak{h}_x$ be a centralizer of $g(x)$, and assume that there is a limit $\mathfrak{h}_{x_i} := \lim_{x \to x_i} \mathfrak{h}_x$ at each singularity x_i.

Let $\mathfrak{g}_x = \mathfrak{h}_x \oplus \sum_\alpha \mathfrak{g}_{\alpha,x}$ be the splitting with respect to the Cartan subalgebra $\mathfrak{h}_x$, and let $e_{\alpha,x}$ be generators of $\mathfrak{g}_{\alpha,x}$.

Then the tangent space $T_g \mathrm{mHiggs}^{\mathrm{fr}}_G(X, D)$ in the equivalence frame $(\xi^L, 0)$ is generated by the sections

$$\xi^L = \sum_i \sum_{\alpha:\langle \alpha, \omega_i^\vee\rangle > 0} \sum_{k_{i,\alpha}=1}^{\langle \alpha, \omega_i^\vee \rangle} e_{\alpha,x_i} \xi_{i,\alpha,k_{i,\alpha}} (x - x_i)^{-\langle \alpha, \omega_i^\vee\rangle}, \qquad \xi_{i,\alpha,k_{i,\alpha}} \in \mathbb{C}$$

whose singular parts at each x_i are in the image of the Ad_g-map. Assuming that $\omega_i^\vee$ is dominant, the singular part is contributed only by the positive roots $\alpha > 0$, and for each positive root α, the degree of pole at x_i is $\langle \check{\omega}_i, \alpha \rangle$. Consequently, the total dimension of the deformation space is

$$\sum_{\alpha > 0} \sum_{x_i \in D} \langle \check{\omega}_{x_i}, \alpha \rangle = 2 \langle \check{\omega}_{tot}, \rho \rangle$$

where Weyl vector ρ is $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ and $\check{\omega}_{tot} = \sum_{x_i \in D} \check{\omega}_{x_i}$.

Given that $\mathrm{mHiggs}_G^{\mathrm{fr}}(X, D) \subset G_1(K(X))$ is a finite-dimensional subvariety, now we'd like to show that this subvariety is a symplectic with respect to Sklyanin's Poisson structure π on $G_1(K(X))$.

Pointwise, a tangent space $T_m S$ to a symplectic leaf $S \subset M$ in a Poisson variety M at a point $m \in M$ is defined to be the image of $\mathrm{Im}\pi$ of the Poisson map $\pi : T^*M \to TM$. The Jacobi identity guarantees that the distribution of subvarieties $\mathrm{Im}\pi \subset T_M$ is integrable.

To a local function (a Hamiltonian) $\phi : M \to \mathbb{C}$ in a neighborhood of $m \in M$ in a Poisson variety M we associate a Hamiltonian vector field ξ_ϕ

$$\xi_\phi = \pi d\phi$$

For local functions ϕ, ψ, Hamiltonian vector fields ξ_ϕ, ξ_ψ belong to a tangent subspace $T_m S$ to a symplectic leaf.

A symplectic structure $\Omega \in \Lambda^2 T^* S$ on S is induced from the Poisson structure π on M by the relation

$$\Omega(X_\psi, X_\phi) = \pi(d\phi, d\psi)$$

We'll start from the Sklyanin's bracket and Poisson structure π on $M = G_1(K(X))$, then

1. show that image of π at $g \in \mathrm{mHiggs}_G^{\mathrm{fr}}(X, D) \subset K(G_1(X))$ is $T_g \mathrm{mHiggs}_G^{\mathrm{fr}}(X, D)$
2. determine the symplectic form $\Omega \in \Lambda^2 T^* \mathrm{mHiggs}_G^{\mathrm{fr}}(X, D)$ such that $\Omega(X_\psi, X_\phi) = \pi(d\phi, d\psi)$ for any local functions on $K(G_1(X))$

Invariance of Ω

We'll start from a formula for Ω on $\mathrm{mHiggs}^{\mathrm{fr}}_G(X; D)$.

Let U_i be small disks around punctures $x_i \in D \cup \{x_\infty\}$. We represent a tangent vector field $\xi \in T_g \mathrm{mHiggs}^{\mathrm{fr}}_G(X, D)$ by a set of sections $(\xi^L, \xi^R)_{(i)}$ regular in each disc U_i, and by a section $(\xi^L, \xi^R)_0$ regular away from D.

Then, for $\xi, \xi' \in T_g \mathrm{mHiggs}^{\mathrm{fr}}_G(X, D)$

$$\Omega(\xi, \xi') = \frac{1}{2\pi\sqrt{-1}} \sum_i \oint_{\partial U_i} \langle \xi_i^L \xi_0^{L\prime} \rangle - \langle \xi_i^R \xi_0^{R\prime} \rangle$$

Notice $\Omega(\xi, \xi')$ that is well defined. Under the change of the equivalence frame in the second argument $(\xi^{L\prime}, \xi^{R\prime}) \to (\xi^{L\prime}, \xi^{R\prime}) + (Ad_g \zeta', -\zeta')$

$$\Omega(\xi, \xi') \to \Omega(\xi, \xi') + \frac{1}{2\pi\sqrt{-1}} \sum_i \oint_{\partial U_i} \langle \xi_i^L, Ad_g \zeta_0' \rangle + \langle \xi_i^R, \zeta_0' \rangle$$

we have

$$\sum_i \oint_{\partial U_i} \langle \xi_i^L, Ad_g \zeta_0' \rangle + \langle \xi_i^R, \zeta_0' \rangle = \sum_i \oint_{\partial U_i} \langle Ad_{g^{-1}} \xi_i^L + \xi_i^R, \zeta_0' \rangle =$$

$$= \sum_i \oint_{\partial U_i} \langle Ad_{g^{-1}} \xi_0^L + \xi_0^R, \zeta_0' \rangle =$$

$$= - \oint_{\partial U_0} \langle Ad_{g^{-1}} \xi_0^L + \xi_0^R, \zeta_0' \rangle = 0$$

and hence Ω is invariant under equivalence in the second argument.

Antisymmetry of Ω

Now, by a suitable choice of equivalence frame we can set $\xi_0^{R\prime} = 0$ and this frame $\xi_i^{L\prime} + Ad_g \xi_i^{R\prime} = \xi_0^{L\prime}$, therefore

$$\Omega(\xi, \xi') = \frac{1}{2\pi\imath} \sum_i \oint_{\partial U_i} \langle \xi_i^L, \xi_i^{L\prime} + Ad_g \xi_i^{R\prime} \rangle = \frac{1}{2\pi\imath} \sum_i \oint_{\partial U_i} \langle \xi_i^L, Ad_g \xi_i^{R\prime} \rangle$$

Alternatively, we can set $\xi_0^{L\prime} = 0$, and in this frame $\xi_0^{R\prime} = \xi_i^{R\prime} + Ad_{g^{-1}} \xi_i^{L\prime}$, hence

$$\Omega(\xi, \xi') = \frac{1}{2\pi\imath} \sum_i \oint_{\partial U_i} \langle \xi_i^R, \xi_i^{R\prime} + Ad_{g^{-1}} \xi_i^{L\prime} \rangle = -\frac{1}{2\pi\imath} \sum_i \oint_{\partial U_i} \langle \xi_i^R, Ad_{g^{-1}} \xi_i^{L\prime} \rangle$$

By comparing the two expressions

$$\Omega(\xi, \xi') = -\Omega(\xi', \xi)$$

Symplectic form on mHiggs

Now we'll demonstrate that the symplectic form Ω on mHiggs $\subset G_1(K(X))$ comes from the Sklyanin's Poisson structure on $G_1(K(X))$

$$\Omega(\xi_{\psi_{x_1}}, \xi_{\phi_{x_2}}) = -\{\psi_{x_1}, \phi_{x_2}\}$$

We can choose the following equivalence frames for the vector fields

$$\xi_{\psi_{x_1}}(w) = (\xi^L_{\psi_{x_1}}, \xi^R_{\psi_{x_2}})_i = \frac{1}{w - x_1}(Ad_{g_{x_1}} \nabla_R \psi_{x_1}, \nabla_R \psi_{x_1})$$

$$\xi_{\phi_{x_2}}(w) = (\xi^L_{\psi_{x_1}}, \xi^R_{\psi_{x_2}})_0 = \frac{1}{w - x_2}(Ad_{g_{x_2}} \nabla_R \phi_{x_2} - Ad_{g_w} \nabla_R \phi_{x_2}, 0)$$

Symplectic form on mHiggs

and evalute $\Omega(\xi_{\psi_{x_1}}, \xi_{\phi_{x_2}})$ from the definition:

$$\Omega(X_{\psi_{x_1}}, X_{\psi_{x_2}}) =$$

$$= \frac{1}{2\pi\imath} \sum_{z_i \in \tilde{D}} \oint_{\partial U_i} \frac{dw}{(w - x_1)(w - x_2)} \langle A_{g_{x_1}} \nabla_R \psi_{x_1}, A_{g_{x_2}} \nabla_R \phi_{x_2} - A_{g_w} \nabla_R \phi_{x_2} \rangle =$$

$$= \frac{-1}{x_1 - x_2} \langle A_{g_{x_1}} \nabla_R \psi_{x_1}, -A_{g_{x_1}} \nabla_R \phi_{x_2} \rangle - \frac{-1}{x_1 - x_2} \langle A_{g_{x_1}} \nabla_R \psi_{x_1}, -A_{g_{x_2}} \nabla_R \phi_{x_2} \rangle =$$

$$= \frac{1}{x_1 - x_2} (\langle \nabla_R \psi_{x_1}, \nabla_R \phi_{x_2} \rangle - \langle \nabla_L \psi_{x_1}, \nabla_L \phi_{x_2} \rangle) = -\{\psi_{x_1}, \phi_{x_2}\}$$

Periodic Monopoles

Now we switch the context from the complex algebraic geometry to real differential geometry and consider a moduli space of solutions to non-linear partial differential equations known as monopole Bogomolny equations defined on a real 3-dimensional Riemannian manifold.

Periodic Monopoles

Take X to be $\mathbb{R}^2 \simeq \mathbb{C}$ with a flat Euclidean metric, and M to be a flat 3d Euclidean manifold $M = X \times S^1$. Lift the divisor $D \subset X$ to $\tilde{D} \subset M$ by taking $m_i = (x_i, s_i)$ with $x_i \in X, t_i \in S^1$.

Let G_c be a maximal compact real form associated to G, let P_c be a principal G_c-bundle on $M \setminus \tilde{D}$ equipped with a smooth connection A and a smooth $\mathfrak{g}$-valued scalar field

$$\phi \in \Gamma(M \setminus \tilde{D}, P \times_{G_c} \mathfrak{g}_c)$$

Periodic Monopoles

Definition

A monopole on M with Dirac singularities at $m_i \in M$ co-weights $\check{\omega}_{m_i}$ is a pair (A, ϕ) of a connection A and adjoint valued scalar field ϕ on M that satisfy

$$F_A = \star D_A \phi$$

such that near each m_i a solution is an image by the co-weight map $\check{\omega}_{m_i} : U(1) \to T_{G_c}$ of the unit $U(1)$ Dirac monopole with singularity at $m \to m_i$.

A unit $U(1)$ Dirac monopole is a local 3d configuration of the form

$$\phi(r) = -\frac{1}{2r}, \qquad F(r) = \frac{1}{2r^2}\mathrm{vol}_{S^2}, \qquad r \to 0$$

where $r = |m - m_i|$ and where vol_{S^2} is a volume form on a two-sphere S^2 centered at $r = 0$. In particular magnetic field has unit flux $\frac{1}{2\pi}\int_{S^2} F = 1$

From monopoles to multiplicative Higgs bundle

Given a monopole on $X \times S^1$, restrict the fields (A, Φ) to a horizontal slice $X \times \{s_0\}$ which does not contain any Dirac singularities.

Then the $(0,1)$ part of the horizontal connection $\bar{\partial}_A$ determines the structure of the holomorphic G-bundle on X.

From monopoles to multiplicative Higgs bundle

Monopole equations imply

$$[\bar{\partial}_A, \nabla_s + \imath\Phi] = 0$$

which in turns imply that in the trivialization in which $\bar{\partial}_A = \bar{\partial}$ the holonomy

$$g(x) := \mathrm{Pexp} \oint_{\{x\} \times S^1} (A_s + \imath\Phi) ds$$

is holomorphic on $X \setminus D$. Hence, we have a holonomy morphism [Kapustin-Cherkis]

$$\mathcal{H} : \mathsf{Mon} \to \mathsf{mHiggs}$$

We define the space of *polystable* multiplicative G-Higgs bundles to be the image of the holonomy map

$$\mathcal{H} : \mathrm{Mon} \twoheadrightarrow \mathrm{mHiggs}^{polystable} \hookrightarrow \mathrm{mHiggs}$$

Then a version of Donaldson-Uhlenbeck-Yau/Kobayashi-Hitchin correspondence provides algebraic description of the stability condition on mHiggs. The idea of this correspondence is to build the reverse map

$$\mathcal{D} : \mathrm{mHiggs}^{polystable} \to \mathrm{Mon}$$

by showing there exists an $S^{1\prime}$-invariant harmonic hermitian metric on an associated holomorphic vector bundle on $X \times S^1 \times S^{1\prime}$ where $S^{1\prime}$ is an auxilary circle used to compactify the self-dual Yang-Mills equations into Bogomolny equations.

Such harmonic metric can be build by running a gradient descent on the Yang-Mills functional: this is a parabolic non-linear partial differential equation (non-linear heat flow).

A hard part of the analysis is to show that the non-linear heat flow actually reaches the harmonic metric in the limit of the infinite time.

For compact complex surfaces the proof was given by Donaldson'83, then in Simpson'88 the heat flow was extended to non-compact surfaces with certain assumption (in particular an assumption of a finite-volume).

Using Simpson's methods, Charbonneau-Hurtubise'08 extended the proof to the situation of monopoles with Dirac singularities on a space $X \times S^1$ where X can be compactified to a finite volume. For example, for CH'08 theorem we can take X to be $\mathbb{P}^1$ of finite folume or an elliptic curve.

Finally, Mochizuki'17 was able to relax the finite volume assumptions of Simpson'88 and consequently constructed the DUY/KH correspondence $\mathcal{D}$ from multiplicative polystable bundles on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ to singular monopoles on $\mathbb{C}^2 \times S^1$

$$\mathcal{D} : \mathrm{mHiggs}^{polystable} \to \mathrm{Mon}$$

The space of singular monopoles Mon on $\mathbb{R}^2 \times S^1_R$ comes with a natural hyperKahler structure induced from the canonical hyperKahler structure on the space of fields (A, Φ) and the monopole equations treated as a hyperKahler moment map for the gauge group action.

A hyperKahler structure can be thought as a twistor ζ-family of ζ-holomorphic $(2, 0)$ symplectic forms $\Omega_{\zeta,R}$.

Given DUY/KH map

$$\mathcal{D} : \mathrm{mHiggs}^{polystable} \to \mathrm{Mon}$$

we can pullback hyperKahler structure $\Omega_{\zeta,R}$ to $\mathrm{mHiggs}^{polystable}$ and define

$$\Omega_{\zeta,R}(\mathrm{mHiggs}) := \mathcal{D}^{*}\Omega_{\zeta,R}(\mathrm{Mon})$$

Next we'll demonstrate the following theorem

Theorem

The holomorphic symplectic structure $\Omega_{\zeta,R}(\mathrm{mHiggs})$ *induced from the hyperKahler structure on the moduli space of monopoles at* $\zeta = 0$ *is equal to the symplectic structure* $\Omega_{Sklyanin}$ *on* $\mathrm{mHiggs} \subset G_1(K(X))$ *obtained from the Sklyanin's Poisson structure on the Poisson-Lie group* $G_1(K(X))$ *with rational* $\mathbf{r}$*-matrix*

Now we compute $\mathcal{D}^*\Omega_{\zeta=0,R}(\mathrm{Mon}((X \times S^1))$ assuming $X = \mathbb{C}^2$.

Let $x, \bar{x}$ be coordinates on X (horizontal), and s be coordinate on S^1 (vertical). Lift monopole on $X \times S^1_s$ to ${S^1}'_t$-translationally invariant self-dual connection A on $X \times Y = X \times S^1_s \times S^{1\prime}_t$ with $A_t \equiv \Phi$. Then take

$$A = A_x dx + A_{\bar{x}} d\bar{x} + A_y dy + A_{\bar{y}} d\bar{y}$$

where $y = s + it$.

The complex part of Bogomolny equations becomes $(0,2)$-flat curvature equation $F^{0,2} = 0$, that is

$$[\bar{\partial}_{\bar{x}} + A_{\bar{x}}, \bar{\partial}_{\bar{y}} + A_{\bar{y}}] = 0$$

A Calabi-Yau holomorphic $(2,0)$-form $v = dx \wedge dy$ on $X \times Y$, together with a Killing form on $\mathfrak{g}$, induces a bilinear $(2,0)$ antisymmetric form on the moduli space of holomorphic G-bundles on $X \times Y$ as follows:

$$\Omega_{\mathrm{Mon}}(\delta A, \delta A') = \int_{X \times Y} v \wedge \operatorname{tr} \delta A \wedge \delta A'$$

equivalently

$$\Omega_{\mathrm{Mon}}(\delta A, \delta A') = \int_{X \times Y} v \wedge \operatorname{tr} \delta A_{0,1} \wedge \delta A'_{0,1}$$

since $\delta A_{1,0}, \delta_{1,0} A'$ variations are projected by $\wedge v$, and the integrand is $(2,2)$ form.

Our goal is to show that

$$\Omega_{\mathrm{Mon}}(\delta A, \delta A') = \int_{X \times Y} v \wedge \operatorname{tr} \delta A_{0,1} \wedge \delta A'_{0,1}$$

on Mon coincides with

$$\Omega_{\mathrm{mHiggs}}(\xi, \xi') = \frac{1}{2\pi\sqrt{-1}} \sum_i \oint_{\partial U_i} \langle \xi_i^L \xi_0^{L\prime} \rangle - \langle \xi_i^R \xi_0^{R\prime} \rangle$$

on mHiggs.

First we choose a cover of X by small discs U_i enclosing the singularities x_i and possibly x_∞, and let $U_0 = X \setminus \bigcup_i U_i$.

Since connections A defining Mon satisfy $F^{0,2} = 0$, it holds for a variation $\delta A_{0,1}$ that $\bar{\partial}_A \delta A_{0,1} = 0$, and consequently in each local chart $U_i \times (0, 2\pi R)$ there is a potential $b^{(i)}$ such that

$$\delta A^{(i)}_{0,1} = \bar{\partial}_A b^{(i)}$$

Moreover, assume that holomorphic G-bundle on $X \times \{0\} = X \times \{2\pi R\}$ does not have deformations (for example trivial). Then, by local gauge transformations we can choose $b^{(i)}$ such that $\bar{\partial}_A b^{(i)}|_{s=0} = \bar{\partial}_A b^{(i)}|_{s=2\pi R} = 0$, that is $\bar{\partial}_A = \bar{\partial}$ when restricted to $s = 0$ or $s = 2\pi R$.

Since $b^{(i)}$ generate local gauge transformations, the variation of the group Higgs field defined by the monodromy $g(x) = Pexp \oint_{\{x\}\times S_1} \bar{A}_{\bar{y}} d\bar{y}$ is

$$\delta g(x) = \xi^L(x) g(x) + g(x)\xi^R(x)$$

where

$$\xi^L(x) = b(x, 0) \qquad \xi^R(x) = b(x, 2\pi R)$$

It remains to see if we can integrate

$$\Omega_{\mathrm{Mon}}(\delta A, \delta A') = \int_{X\times Y} v \wedge \operatorname{tr} \bar{\partial}_A b \wedge \bar{\partial}_A b'$$

and express the result in terms of the boundary terms $(\xi^L, \xi^R)^{(i)}$.

Since $\bar{\partial}_A^2 = 0$ it holds that $v \wedge \operatorname{tr} \bar{\partial}_A b \wedge \bar{\partial}_A b' = d(v \wedge b \bar{\partial}_A b')$ and we integrate to the boundary of each U_i by the Stokes formula

$$\Omega_{\mathrm{Mon}}(\delta A, \delta A') = \sum_i \int_{\partial U_i \times (0, 2\pi R) \times S_t^{1\prime}} v \wedge b \bar{\partial}_A b'$$

There is no contribution from the bases of cylinders $U_i \times \{0\}, U_i \times \{2\pi R\}$ because of our choise $\bar{\partial}_A b = 0$ at $s = 0, 2\pi R$.

Because of our choice of the cover, f $\partial U_0 = -\sum_{i \neq 0} \partial U_i$, and consequently

$$\Omega_{\mathrm{Mon}}(\delta A, \delta A') = \sum_{i \neq 0} \int_{\partial U_i \times (0, 2\pi R) \times S_t^{1\prime}} v \wedge (b_i - b_0) \bar{\partial}_A b_i'$$

where we used that on the boundary ∂U_i there is agreement in variations $\bar{\partial}_A b_i' = \bar{\partial}_A b_0'$.

Now we can integrate by parts once again since $\bar{\partial}_A b_i = \bar{\partial}_A b_0$

$$(b_i - b_0)\bar{\partial}_A b_i' = \bar{\partial}_A((b_i - b_0)b_i')$$

from

$$\Omega_{\mathrm{Mon}}(\delta A, \delta A') = \sum_{i \neq 0} \int_{\partial U_i \times (0, 2\pi R) \times S_t^1{}'} v \wedge (b_i - b_0)\bar{\partial}_A b_i'$$

we obtain by Stokes

$$\Omega_{\mathrm{Mon}}(\delta A = \bar{\partial}_A b, \delta A' = \bar{\partial}_A b') = \sum_{i \neq 0} \int_{\partial U_i} dz(\operatorname{tr} \xi_0^L \xi_i^{L\prime} - \operatorname{tr} \xi_0^R \xi_i^{R\prime})$$

which exactly matches the formula for Ω(mHiggs) obtained by the restriction of the Sklyanin's rational **r**-matrix Poisson structure on the symplectic leaf mHiggs in the rational Poisson-Lie group $G_1(K(X))$.

To summarise, we've shown that

- $\mathrm{mHiggs}_G^{\mathrm{fr},\mathit{polystable}}(X, D)$ is a symplectic leaf in rational Poisson-Lie loop group $G_1(K(X))$ and carries Sklyanin holomorphic symplectic structure determined by the rational $\mathbf{r}$-matrix
- Sklyanin holomorphic symplectic structure on $\mathrm{mHiggs}_G^{\mathrm{fr},\mathit{polystable}}(X, D)$ matches the $\zeta = 0$ hyperKahler structure on $\mathrm{Mon}(X \times S^1)$ under the Donaldson-Uhlenbeck-Yau / Kobayashi-Hitchin / Cherkis-Kapustin correspondence between periodic monopoles and multiplicative Higgs bundles.

Closely connected works are

- Gerasimov-Kharchev-Lebedev-Oblezin'05 on Yangian and quantized singular monopoles on $\mathbb{R}^3$
- Kamnitzer-Webster-Weekes-Yacobi'14 on Yangian and slices in affine Grassmanian
- Braverman-Finkelberg-Nakajima'16 on Coulomb branches and slices in affine Grassmanian

ϵ-twist

Now, for an automorphism ϵ of the curve X we can consider the moduli space of monopoles on the twisted product $M_\epsilon = X \times_\epsilon S^1$, where X is fibered over S^1 such that there is a monodromy ϵ when X is moved around the circle S^1.

If X is flat (for example $X = \mathbb{R}^2$) and ϵ is an isometry (for example a constant shift $x \to x + \epsilon$), the resulting moduli space $\mathrm{Mon}(X \times_\epsilon S^1)$ carries hyperKahler metric induced from the flat Euclidean structure on $X \times_\epsilon S^1$

In the complex structure $\zeta = 0$, a complex part of the Bogomolny equations defines a parallel transport from $s = 0$ to $s = 2\pi R$ with an extra shift by ϵ along x. Consequently, we have a map

$$\mathsf{Mon}_{G_c}(X \times_\epsilon S^1) \to \mathrm{Conn}^\epsilon_G(X)$$

where $\mathrm{Conn}^\epsilon_G(X)$ denotes the moduli space of pairs (principal G-bundle P on X, ϵ-difference connection on P).

The reverse map can be computed again by the Donaldson technique: running the heat flow equation on a Hermitian metric in a holomorphic G-bundle until the metric becomes harmonic [Mochizuki'17]

The space $\mathrm{Conn}^\epsilon_G(X)$ is a difference version of the moduli space of flat connections on X that is usually treated in the "de Rham" picture of geometric Langlands.

Steinberg section

The symplectic space $\mathrm{mHiggs}_G^{\mathrm{fr}}(X, D)$ comes with a structure of the fibration of algebraic integrable system

$$\mathrm{mHiggs}_G^{\mathrm{fr}}(X, D) \xrightarrow{\chi} B$$

induces from the adjoint invarint projection (Chevalley)

$$G_1[[x^{-1}]] \xrightarrow{\chi} T_1[[x^{-1}]]/W$$

where T is the maximal torus in G and W is the Weyl group. Typically (if G is simple simply-connected, for example), χ is generated by the characters of the highest weight fundamental representations.

The projection χ comes with a natural section s constructed by Steinberg'65

$$T_1[[x^{-1}]]/W \to G_1[[x^{-1}]]$$

ϵ-difference opers

- Under the ϵ-twist which deforms $\mathrm{mHiggs}_G(X, D)$ to $\mathrm{Conn}^{\epsilon}_G(X, D)$, the Steinberg section becomes a half-dimensional subvariety of $\mathrm{Conn}^{\epsilon}_G(X, D)$ that is a finite-difference analog of the brane of opers. The brane of opers also appears as a mirror of the canonical coisotropic brane [Kapustin-Witten'06, Nekrasov-Witten'10] quantizing moduli space of Higgs bundles.
- The quasi-diagonalization of ϵ-difference opers yield the Bethe ansatz like equations for the spectrum of the quantized integrable system built on multiplicative G-Higgs bundles (see e.g. Frenkel-Reshetikhin'98, Sevostyanov'98, Nekrasov-Pestun-Shatashvili'13, String-Math'17 talk, and Koroteev-Sage-Zeitlin'18)
- Futhermore, $\hbar$-quantization of ϵ-difference opers yields the difference $(\epsilon, \hbar)$-W-algebra whose currents are also known as $(\epsilon, \hbar)$-characters Nekrasov'15, Kimura-Pestun'15.

Remarks

- A moduli space multiplicative G-Higgs bundles for G of ADE type is identified with Coulomb branches of $\mathcal{N} = 2$ ADE quiver gauge theories Cherkis-Kapustin, Nekrasov-Pestun, Braverman-Finkelberg-Nakajima
- SYZ duality on the fibers of the integrable system of multiplicative Higgs bundles provides geometric view on the q-geometric Langlands Frenkel-Reshetikhin, Aganagic-Frenkel-Okounkov
- Quantization of mHiggs in complex structure $\zeta = 0$ provides Yangian modules and various versions of spinchains and analytic Bethe-ansatz
- $\hbar$-Quantization of Steinberg section in complex structure ϵ leads to $\hbar, \epsilon$-Walgebras and that relates to Vertex algebra at the corner Gaiotto-Rapcak, and COHA of Kontsevich-Soibelman Rapcak, Soibelman, Yang, Zhao
- Quantization of mHiggs by the path integral of $\oint pdq$ leads to the semi-holomorphic Cherns-Simons theory on $X \times S^1 \times \mathbb{R}^1_t$ Nekrasov'96, Costello-Witten-Yamazaki'17 in the same way as quantization of flat connections on X leads to Chern-Simons on $X \times \mathbb{R}^1$ Witten'89

Thank you!