Lie, associative and commutative quasi-isomorphism

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Warmup

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The commutator of two primitive elements in a bialgebra is again primitive, so the set of primitive elements is naturally a Lie algebra.
PBW $\implies$ The natural map $\mathfrak{g} \to U\mathfrak{g}$ takes $\mathfrak{g}$ isomorphically onto the Lie algebra of primitives in $U\mathfrak{g}$.
Hence the answer is YES.

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More formally: if $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras such that $U\mathfrak{g}$ and $U\mathfrak{h}$ are isomorphic as associative algebras, must $\mathfrak{g}$ and $\mathfrak{h}$ be isomorphic?

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Spoiler: The answer is still YES!

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We say that A and A' are quasi-isomorphic as commutative dg algebras, and write $A \simeq A'$, if there exists a zig-zag of commutative dg algebras and quasi-isomorphisms between them:

$$A \xleftarrow{\sim} \bullet \xrightarrow{\sim} \cdots \xleftarrow{\sim} \bullet \xrightarrow{\sim} A'.$$

CAUTION: This does not imply the existence of a quasi-isomorphism $A \to A'$!

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A priori it could happen that two commutative dg algebras are quasi-isomorphic as associative dg algebras, but not quasi-isomorphic as commutative dg algebras. This would mean that there exists a zig-zag of quasi-isomorphisms

$$A \xleftarrow{\sim} \bullet \xrightarrow{\sim} \cdots \xleftarrow{\sim} \bullet \xrightarrow{\sim} A'$$

in the category of associative dg algebras, but no zig-zag in which every intermediate dg algebra is actually commutative.

First main theorem

Theorem A (C-P-RN-W '19): Let A and A' be commutative dg algebras. If A and A' are quasi-isomorphic as dg algebras, then they are also quasi-isomorphic as commutative dg algebras.

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Koszul duality interchanges the forgetful functor from commutative to associative algebras (on the right hand side) and the universal enveloping functor from Lie algebras to associative algebras (on the left hand side).

Second main theorem

Theorem B (C-P-RN-W '19): Let $\mathfrak{g}$ and $\mathfrak{g}'$ be dg Lie algebras concentrated in positive or negative homological degree. If $U\mathfrak{g}$ and $U\mathfrak{g}'$ are quasi-isomorphic as dg algebras, then $\mathfrak{g}$ and $\mathfrak{g}'$ are quasi-isomorphic.

Theorem B (C-P-RN-W '19): Let $\mathfrak{g}$ and $\mathfrak{g}'$ be dg Lie algebras concentrated in positive or negative homological degree. If $U\mathfrak{g}$ and $U\mathfrak{g}'$ are quasi-isomorphic as dg algebras, then $\mathfrak{g}$ and $\mathfrak{g}'$ are quasi-isomorphic.

Remark: We need the assumption on positive or negative grading because Koszul duality is rarely a perfect duality — information is usually lost when passing from one side of the duality to the other. This version of Theorem B does not resolve the version of the question for classical Lie algebras (i.e. with no grading or differential).

Second main theorem, variant

Theorem B, variant: Let $\mathfrak{k}$ and $\mathfrak{k}'$ be dg Lie coalgebras. If their universal coenveloping coalgebras $U^c\mathfrak{k}$ and $U^c\mathfrak{k}'$ are weakly equivalent as dg coalgebras, then $\mathfrak{k}$ and $\mathfrak{k}'$ are weakly equivalent as dg Lie coalgebras.

Theorem B, variant: Let $\mathfrak{k}$ and $\mathfrak{k}'$ be dg Lie coalgebras. If their universal coenveloping coalgebras $U^c\mathfrak{k}$ and $U^c\mathfrak{k}'$ are weakly equivalent as dg coalgebras, then $\mathfrak{k}$ and $\mathfrak{k}'$ are weakly equivalent as dg Lie coalgebras.

Corollary: Let $\mathfrak{g}$ and $\mathfrak{g}'$ be finite dimensional Lie algebras. If $U\mathfrak{g}$ and $U\mathfrak{g}'$ are isomorphic, then so are $\mathfrak{g}$ and $\mathfrak{g}'$.

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Corollary: Let $\mathfrak{g}$ and $\mathfrak{g}'$ be finite dimensional Lie algebras. If $U\mathfrak{g}$ and $U\mathfrak{g}'$ are isomorphic, then so are $\mathfrak{g}$ and $\mathfrak{g}'$.

The corollary follows by taking the linear duals of $\mathfrak{g}$ and $\mathfrak{g}'$ to obtain Lie coalgebras. We need finite dimensionality for the linear dual to be a Lie coalgebra.

Remarks

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Question of whether $U\mathfrak{g}$ determines $\mathfrak{g}$ is analogous to more well studied question of whether a group can be recovered from its group algebra. Hertweck '01: there exists finite groups G and H such that $\mathbb{Z}G \cong \mathbb{Z}H$ as associative algebras, but $G \ncong H$.

Applications to rational homotopy theory

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(Theorem A + Sullivan) $\implies$ X and Y have the same rational homotopy type if and only if $C^*(X, \mathbb{Q}) \simeq C^*(Y, \mathbb{Q})$ as dg algebras.

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(Theorem B + Quillen) $\implies$ X and Y have the same rational homotopy type if and only if $C_*(\Omega X, \mathbb{Q}) \simeq C_*(\Omega Y, \mathbb{Q})$ as dg algebras.

A first reformulation

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- $A \simeq A'$ as commutative dg algebras $\iff$ there exists a C_∞-quasi-isomorphism $A \to A'$.

Theorem A $\iff$ If there exists an A_∞-quasi-isomorphism between two C_∞-algebras, then there also exists a C_∞-quasi-isomorphism between them.

Recall that an A_∞-algebra A is equipped with operations $\mu_n \colon A^{\otimes n} \to A$, $n \geq 1$, so that μ_1 is a differential, and μ_2 is a multiplication which is associative up to higher homotopies provided by the μ_n, $n \geq 3$.

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For example, μ_2 vanishes on all elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x$, i.e. the multiplication is commutative. The condition that the higher μ_n vanish on shuffles is some kind of commutativity condition on the homotopies.

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An A_∞-morphism of A_∞-algebras $f\colon A \to B$ is specified by a collection of maps $f_n\colon A^{\otimes n} \to B$, $n \geq 1$. If A and B are C_∞-algebras, then f is called a C_∞-morphism if the components f_n similarly vanish on signed shuffles.

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In particular, if A and B are C_∞-algebras, then there are typically many more A_∞-morphisms $A \to B$ than there are C_∞-morphisms.

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Theorem A $\iff$ Given two C_∞-algebra structures on the same graded vector space, and an A_∞-isomorphism between them whose linear term is the identity map (an A_∞-isotopy), there also exists a C_∞-isotopy between them.

Interlude: Formal deformation theory

Principle due to Deligne, Drinfeld, and developed by Feigin, Hinich, Kontsevich–Soibelman, Lurie, Pridham (and others): every dg Lie algebra gives rise to a formal deformation problem and every formal deformation problem arises from a dg Lie algebra.

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If $\mathfrak{g}$ is a dg Lie algebra, then the solutions to the deformation problem are the Maurer–Cartan elements

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Deformation equivalence of solutions is defined by the action of the gauge group $\exp(\mathfrak{g}_0)$.

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Our setting here is somewhat different: we will consider complete dg Lie algebras, i.e. dg Lie algebras equipped with a complete filtration which makes the required power series converge. No Artin rings anywhere.

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There is a dg Lie subalgebra $\mathrm{Def}_{C_\infty}(V) \subset \mathrm{Def}_{A_\infty}(V)$ parametrizing C_∞-algebra structures on V.

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Similarly $\mathrm{Def}_{C_\infty}(V) \approx$ Harrison cochains

A third reformulation

Question: Let $i\colon \mathfrak{h} \hookrightarrow \mathfrak{g}$ be a dg Lie subalgebra. When do we have an inclusion

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Theorem (C-P-RN-W '19) If $i\colon \mathfrak{h} \hookrightarrow \mathfrak{g}$ is an inclusion of complete dg Lie algebras and there exists $r\colon \mathfrak{g} \to \mathfrak{h}$ which is a filtered retraction of $\mathfrak{h}$-modules, i.e.

- $r \circ i = \mathrm{id}_{\mathfrak{h}}$
- $r[i(x), y] = [x, r(y)]$

then $\mathrm{MC}(\mathfrak{h})/\exp(\mathfrak{h}_0) \longrightarrow \mathrm{MC}(\mathfrak{g})/\exp(\mathfrak{g}_0)$ is injective.

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This proves Theorem A.

A proof sketch

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Define inductively elements $a^{(n)} \in \mathfrak{g}_0$ and $x^{(n)} \in \mathrm{MC}(\mathfrak{h})$ by

$$\begin{aligned} a^{(0)} &= a & a^{(n+1)} &= \mathrm{BCH}(a^{(n)}, -r(a^{(n)})) \\ x^{(0)} &= x & x^{(n+1)} &= \exp(r(a^{(n)})) \cdot x^{(n)} \end{aligned}$$

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One shows $\lim_{n\to\infty} x^{(n)} = y$ and $\lim_{n\to\infty} r(a^{(n)}) = 0$

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So $\prod_{n\geq 0} \exp(r(a^{(n)})) \in \exp(\mathfrak{h}_0)$ is a well defined gauge from x to y. **QED**

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But $U\mathcal{L}(-) \cong \Omega(-)$. So $\Omega\mathcal{C}\mathfrak{g} \simeq \Omega\mathcal{C}\mathfrak{g}'$.

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But $U\mathcal{L}(-) \cong \Omega(-)$. So $\Omega\mathcal{C}\mathfrak{g} \simeq \Omega\mathcal{C}\mathfrak{g}'$.

Thus we get $\mathcal{C}\mathfrak{g} \simeq B\Omega\mathcal{C}\mathfrak{g} \simeq B\Omega\mathcal{C}\mathfrak{g}' \simeq \mathcal{C}\mathfrak{g}'$, zig-zag of quasi-isomorphisms of dg coalgebras.

Naive proof of Theorem B from Theorem A

Let B, Ω and $\mathcal{C}$, $\mathcal{L}$ denote the bar and cobar functors between dg algebras and dg coalgebras, and dg Lie algebras and cocommutative dg coalgebras, respectively.

Suppose $U\mathfrak{g} \simeq U\mathfrak{g}'$. Then $U\mathcal{L}\mathcal{C}\mathfrak{g} \simeq U\mathcal{L}\mathcal{C}\mathfrak{g}'$.

But $U\mathcal{L}(-) \cong \Omega(-)$. So $\Omega\mathcal{C}\mathfrak{g} \simeq \Omega\mathcal{C}\mathfrak{g}'$.

Thus we get $\mathcal{C}\mathfrak{g} \simeq B\Omega\mathcal{C}\mathfrak{g} \simeq B\Omega\mathcal{C}\mathfrak{g}' \simeq \mathcal{C}\mathfrak{g}'$, zig-zag of quasi-isomorphisms of dg coalgebras.

By a dual of form of Theorem A we obtain that $\mathcal{C}\mathfrak{g} \simeq \mathcal{C}\mathfrak{g}'$ as cocommutative dg coalgebras. Hence $\mathfrak{g} \simeq \mathcal{L}\mathcal{C}\mathfrak{g} \simeq \mathcal{L}\mathcal{C}\mathfrak{g}' \simeq \mathfrak{g}'$.

Tack!