5d SCFTs, Flavors and BPS States

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1906.11820 and (to appear soon)$^2$, with
Fabio Apruzzi, Craig Lawrie, Ling Lin, Yi-Nan Wang
(Oxford/UPenn)
5d $\mathcal{N} = 1$ Gauge Theories and SCFTs

5d $\mathcal{N} = 1$ SCFTs are intrinsically non-perturbative quantum field theories. At low energies: there can be an effective descriptions in terms of weakly coupled gauge theories.

5d $\mathcal{N} = 1$ SUSY massless multiplets:

- Vector multiplet: $A = (A_\mu, \phi^i, \lambda)$
- Hyper-multiplet: $h = (h \oplus h^c, \psi)$

Coulomb branch: $\langle \phi^i \rangle \neq 0$.

Example: [Seiberg][Morrison, Seiberg]

$G_{\text{gauge}} = SU(2)$ with $N_F$. Coulomb branch is 1d ("rank 1") and weakly couple flavor symmetry $SO(2N_F)$ enhances to $E_{N_F+1}$ in the UV.

In general: difficult to come by properties of the UV fixed points.
Motivation

• Classification of 6d $(1,0)$ SCFTs: F-theory on elliptically fibered Calabi-Yau threefolds (CY3). [Heckman, Morrison, (Rudelius), Vafa][Bhardwaj]

• 6d to 5d:
  ⇒ Compactification on $S^1$: 5d marginal theory (UV completes in 6d).
  ⇒ Add masses to flavors: flows to a 5d UV fixed point.

• Can we identify and characterize all such 5d SCFTs?
  ⇒ Yes and we propose a systematic, combinatorial way to do so

• Whether these are all 5d SCFTs remains to be seen (hinges on the classification of canonical singularities in CY3, and whether these are all obtained from elliptic models). See [Xie, Yau]
Proposal

[Apruzzi, Lawrie, Lin, SSN, Wang]

# 5d SCFTs from M-theory on elliptic CY3 with non-flat fibers

# Extended Coulomb branch phases ($\langle \phi \rangle$ + Coulomb branch parameters to weakly gauge the classical flavors) are matched with geometric moduli space (resolutions of singular elliptic fibrations)

# We find a systematic, combinatorial description in terms of graphs (“Combined Fiber Diagrams” (CFDs)) which describe 5d SCFTs including

- strongly coupled (usually enhanced) flavor symmetry
- Mass deformations, i.e. descendant SCFTs
- BPS states
General Strategy: 6d to 5d

5d N=1 SCFT
$G_F^{(5d)}$

5d Gauge Theory
$G_{\text{gauge}}$
$G_{F,\text{classical}}^{(5d)}$

5d Marginal Theory
$G_F^{(5d)}$

RG-flow to UV

6d (0,1) on Tensor Branch
$G_F^{(6d)}$

Masses $m_f$

6d (0,1) SCFT
$G_F^{(6d)}$

$S^1$

$S^1$+holonomies

$<\phi>$
Refined Strategy

- M/ canonical singularity CY$_3$ $G_F^{(5d)}$
  - Partial Singular limit
  - M/ smooth CY$_3$ Non-flat Fibers $S_i$ $G_F^{(5d)} < S_i$
  - Non-flat Fiber Resolution

- F/ smooth CY$_3'$ resolved base
  - Base Blowup

- M/ singular CY$_3$ nonminimal $(g_1, g_2)$

- F/ elliptic CY$_3$ non-minimal singularity $(g_1, g_2)$

[Apruzzi, Lawrie, Lin, SSN, Wang]
6d to 5d SCFT

# Earlier developments: [Intrilligator, Morrison, Seiberg][Klemm, Mayr, Vafa][Aharony, Hanany, Kol]
# Recently, different approach from 6d SCFT on $S^1$: [Del Zotto, Heckman, Morrison][Jefferson, Katz, Kim, Vafa][Bhardwaj, Jefferson][Apruzzi, Lin, Mayrhofer][Closset, Del Zotto, Saxena]. The strategy so far:

- **Tensor branch of 6d on $S^1$** (base blowup)
- To reach SCFT usually need to **flop** surfaces in the CY3 
  ⇒ does not retain elliptic fibration structure
- **SCFT enhanced flavor symmetries** not manifest in geometry

# Main new idea [ALLSW]: non-flat resolution of the elliptic fibration, retains info about flavor symmetry, and even BPS states! Succinct description in terms of simple graphs “CFDs”
6d SCFTs from F-theory on CY3

F-theory on non-compact CY3:

- Elliptic CY3 $E_\tau \hookrightarrow Y_3 \to B$, with a section has Weierstrass form
  \[ y^2 = x^3 + fx + g, \quad f, g \in H^0(K_B^{4/6}). \]

  Noncompact base $B$: $\mathbb{C}^2$, coordinates $u, v$

- Discriminant: Singular fiber above $u = 0$:
  \[ \Delta = 4f^3 + 27g^2 = O(u^n) \]

- Kodaira fiber above $u = 0$, i.e. in codim 1.
Classification of Singular Fibers

Codim 1 in base: Kodaira classified singular fibers

\[
\text{Singular fibers} \quad \leftrightarrow \quad (\text{Decorated}) \text{ ADE affine Dynkin diagram}
\]
Kodaira’s classification of singular fibers and gauge groups

<table>
<thead>
<tr>
<th></th>
<th>ord$_S$(f)</th>
<th>ord$_S$(g)</th>
<th>ord$_S$(Δ)</th>
<th>singularity</th>
<th>local gauge group factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_0$</td>
<td>≥ 0</td>
<td>≥ 0</td>
<td>0</td>
<td>none</td>
<td>–</td>
</tr>
<tr>
<td>$I_1$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>none</td>
<td>–</td>
</tr>
<tr>
<td>$I_2$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>$A_1$</td>
<td>$SU(2)$</td>
</tr>
<tr>
<td>$I_m, m \geq 1$</td>
<td>0</td>
<td>0</td>
<td>$m$</td>
<td>$A_{m-1}$</td>
<td>$Sp([\frac{m}{2}])$ or $SU(m)$</td>
</tr>
<tr>
<td>$II$</td>
<td>≥ 1</td>
<td>1</td>
<td>2</td>
<td>none</td>
<td>–</td>
</tr>
<tr>
<td>$III$</td>
<td>1</td>
<td>≥ 2</td>
<td>3</td>
<td>$A_1$</td>
<td>$SU(2)$</td>
</tr>
<tr>
<td>$IV$</td>
<td>≥ 2</td>
<td>2</td>
<td>4</td>
<td>$A_2$</td>
<td>$Sp(1)$ or $SU(3)$</td>
</tr>
<tr>
<td>$I^*_0$</td>
<td>≥ 2</td>
<td>≥ 3</td>
<td>6</td>
<td>$D_4$</td>
<td>$G_2$ or $SO(7)$ or $SO(8)$</td>
</tr>
<tr>
<td>$I^*_m, m \geq 1$</td>
<td>2</td>
<td>3</td>
<td>$m + 6$</td>
<td>$D_{m+4}$</td>
<td>$SO(2m + 7)$ or $SO(2m + 8)$</td>
</tr>
<tr>
<td>$IV^*$</td>
<td>≥ 3</td>
<td>4</td>
<td>8</td>
<td>$E_6$</td>
<td>$F_4$ or $E_6$</td>
</tr>
<tr>
<td>$III^*$</td>
<td>3</td>
<td>≥ 5</td>
<td>9</td>
<td>$E_7$</td>
<td>$E_7$</td>
</tr>
<tr>
<td>$II^*$</td>
<td>≥ 4</td>
<td>5</td>
<td>10</td>
<td>$E_8$</td>
<td>$E_8$</td>
</tr>
<tr>
<td>non-minimal</td>
<td>≥ 4</td>
<td>≥ 6</td>
<td>≥ 12</td>
<td>non-canonical</td>
<td>–</td>
</tr>
</tbody>
</table>
Minimal versus non-minimal: codimension 2

In codimension 2: \( u = v = 0 \) in the base:

1. **Minimal:** ordinary bifundamental matter, and codimension two fiber are (monodromy-reduced) Kodaira fibers (collection of rational curves).

2. **Non-minimal:** Weierstrass model

   \[ y^2 = x^3 + fx + g \], \quad \text{ord}_{u=v=0}(f, g, \Delta) \geq (4, 6, 12)

   no crepant resolution of fiber possible that keeps complex 1d fiber.

   Two options:

   (1) Blowup the base (6d approach)

   (2) Blowup fiber with non-flat fibers
Non-minimal Singularities: (1) Base-blowup

**Blowup the base:** \( u = v = 0 \) non-minimal point \( p \): blowup by bluing in \( \mathbb{P}^1 \).
\( \Sigma_i \cong \mathbb{P}^1 \) with \( \Sigma_i^2 = n_i \) (degree of the normal bundle in \( B \))

\[
\begin{array}{c}
\text{n}_1 & \text{n}_2 \\
\longrightarrow & \longrightarrow \\
\text{n}_1+1 & \text{n}_2+1
\end{array}
\]

Example: Conformal Matter \[\text{[del Zotto, Heckman, Tomasiello, Vafa]}\]

Start with singularities \( G_1 - G_2 \) colliding at a point, e.g.

\[(E_6, E_6) : \quad y^2 = x^3 + u^4 v^4\]

Blowup the point \( u = v = 0 \) repeatedly, until model is minimal:

\[ [E_6](-1)(-3)(-1)[E_6] \]

Has \( E_6^2 \) flavor symmetry and an \( su(3) \) gauge group on the \(-3\) curve.
Non-minimal Singularities: (2) Non-Flat Fibers

Non-flat fiber resolution: crepant resolution, but includes complex surface components $S_i$, in addition to rational curves:

Physics:
Non-flat fiber surfaces are compact $S_i \leftrightarrow U(1)$ gauge fields.
Flavor $\mathbb{P}^1$s contained in $S_i$ remain flavor symmetries as $\text{vol}(S_i) \to 0$ limit.
[Intriligator, Morrison, Seiberg]

New insight: for non-flat fibration: $S_i$ are non-flat fiber components

1. $S_i$ compact divisors $i = 1, \cdots, r = \text{rank} \Rightarrow U(1)^r$ gauge bosons

2. $S_i \to C_i$ (collapse to curve) $\Rightarrow$ enhancement of $U(1)^r \subset G_{\text{gauge}}$

3. $S_i \to \text{point}$ (collapse to point) $\Rightarrow$ strong coupling

4. Coulomb branch including hypers in $R_F$ of $G_{F, \text{classical}}^{5d}$:

$$\mathcal{F}_{1\text{-loop}} = \frac{1}{12} \left( \sum_\alpha |\phi \cdot \alpha|^3 - \sum_{\lambda_F \in R_F} |\lambda_F \cdot \phi + m_F|^3 \right)$$

$\Rightarrow$ Extended Kähler cone: $m_F = \text{mass deformations, or CB parameters for weakly gauging flavor symmetry.}$

5. **Flavor symmetry**: determined by ADE singularities over non-compact curves, whose fibral curves are contained in $S_i$

6. M2-wrapping modes on curves: BPS states
Example: The E-string and 5d rank 1 SCFTs

1. Starting point 6d: non-compact base with coordinates \( u, v \): \( E_8 - I_1 \) with Tate model: \( \text{ord}_u(b_i) = (1, 2, 3, 4, 5) \) and \( \text{ord}_v(b_i) = (0, 0, 0, 0, 1) \)

\[
y^2 + b_1 u x y + b_3 u^3 = x^3 + b_2 u^2 x^2 + b_4 u^4 x + b_6 u^5 v
\]

2. Non-flat Resolution: sequence of blowup

1. \( \mathbb{P}^1_i \hookrightarrow D_i \rightarrow (u = 0) \): non-compact divisors

\( D_0, D_1, \cdots, D_8 \leftrightarrow \) Affine roots of \( E_8 \)

2. \( S_1 \leftrightarrow \) Non-flat fiber, i.e. surface

\( \Rightarrow \) Cartan of rank 1 gauge group

3. Different resolutions: \( S_1 \) contain different subset of \( \mathbb{P}^1_i \)
Geometrically:

- **Different resolution sequences** of the $E_8$ and codim 2 non-flat locus, yield different 5d theories.

- They are related by **flops**: shrinking $-1$ curves and transforming them out of the surface $S_1$.

- **Strategy**: start with “marginal” theory, i.e. $\mathbb{P}^1_i \subset S_1$ for all $i = 0, \cdots, 8$, and descend to other models by flops.

Marginal model for rank one theories: $S_1 = gdP_9$ showing curves $\mathbb{P}^1_i = S_1 \cdot D_i$

This is the fiber for the marginal 5d theory. We can proceed by computing other blowups... or be more efficient.
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Non-flat Resolutions as Graph-Transitions

Start with diagram of curves that are contained in non-flat fiber $S_1$. 

*For higher rank: Consider Mori cone generators of reducible surface $\cup_k S_k$.*

**Combined Fiber Diagram (CFD):**

CFDs are graphs with:

- # Vertices: Curves $C_i$, labeled by $C_i^2 = n_i$
- # Edges: $C_i \cdot C_j = m_{i,j}$ edges connecting vertices
- # Marked vertices: $n_i = -2$ colored
  - $\Rightarrow$ subgraph = Dynkin of superconformal flavor symmetry

**Marginal rank 1 CFD:**

```
\begin{center}
\begin{tikzpicture}
\node (n1) at (0,0) {$-1$};
\node (n2) at (1,0) {$2$};
\node (n3) at (2,0) {$2$};
\node (n4) at (3,0) {$2$};
\node (n5) at (4,0) {$2$};
\node (n6) at (5,0) {$2$};
\node (n7) at (6,0) {$2$};
\node (n8) at (7,0) {$2$};
\node (n9) at (8,0) {$2$};
\node (n10) at (9,0) {$2$};
\node (n11) at (10,0) {$2$};
\end{tikzpicture}
\end{center}
```
CFD Transitions

Given a CFD, the descendant CFDs are obtained by removing \( n_i = (-1) \) vertex and updating

\[
\begin{align*}
n'_j &= n_j + m^2_{i,j} \\
m'_{j,k} &= m_{j,k} + m_{i,j}m_{i,k}
\end{align*}
\]

Any \((-2)\)-vertex whose \( n_j \) changes becomes unmarked.

\[ \Rightarrow \] Network of CFDs/SCFTs

What are these?
# 5d SCFT: gives mass to flavors and triggers RG-flow to another SCFT
# Geometry: \((-1)\) curves can be contracted and flopped out of \( S_i \):

\[ \text{Diagram:} \]

\[ \text{Diagram:} \]
\[ GF = E_8 \]
\( SU(3) \times SU(2) \)
Rank 1 CFDs

This constructs precisely the known theories:

☆ $SU(2)$ gauge theory with $N_F$ flavors, which enhances to $E_{N_F+1}$ flavor symmetry at SCFT point
  ⇒ this reproduces known facts about rank 1 [Seiberg]

☆ From non-flat fiber: $\mathbb{P}^1$s that are contained in $S_1$ (green) encode superconformal flavor symmetry $G_F$

☆ Includes 5d SCFT without weakly coupled gauge theory description, geometry of $S_1 = \mathbb{P}^2$ (no ruling)
What about higher rank? Rank 2 ✓

Rank 2 theories:
geometric classification was recently obtained in [Jefferson, Katz, Kim, Vafa],
and from 5-brane webs by [Hayashi, Kim, Lee, Yagi].

This network of SCFTs can be reconstructed from CFDs, in addition to the
full flavor symmetry, and some BPS states.

Strategy: determine the marginal theories and apply CFD-transitions.
A flavor of Non-flat Resolutions: Rank 2 E-string

Codimension two collision of $E_8 - I_2$:

$SU(6) \times U(1)$

$SO(12) \times U(1)$
Generating all Rank 2 Theories

Rank 2 theories: marginal theories have CFDs

\[ E_8 \times SU(2) \]

\[ D_5 - D_5 \]

\[ SU(3) \text{ on a } (-1)\text{-curve} + 12 \text{ hypers} \]

\[ SU(3) \text{ on a } (-2)\text{-curve} + 6 \text{ hypers} \]
blue: only $D_{10}$ realization; green: also rank 2 E-string realization; levels: $SU(3)$; pink: $SU(2)^2$; grey: no weakly-coupled gauge theory realization.
$SU(3)$ on a $(-1)$-curve + 12 hypers
$SU(3)$ on a $(-2)$-curve + 6 hypers
Cross-checks:

- Rank 1/2: complete agreement with expected network and enhanced flavor symmetry.

- Only resolution necessary to determine the CFD of the marginal theory.

- Cross-checks 1: Explicit geometric non-flat resolutions confirms the models in low rank (using methods from [Lawrie, SSN])

- Cross-checks 2: Models with weakly coupled gauge theory description [Apruzzi, Lawrie, Lin, SSN, Wang, to appear Part I] reconstruct all fibers (as well as all gauge theory phases) from systematic analysis of the Coulomb branch using methods of [Hayashi, Lawrie, Morrison SSN]: for rank 2: $SU(3)$, $Sp(2)$ and $SU(2) \times SU(2)$ gauge theory descriptions. [Apruzzi, Lawrie, Lin, SSN, Wang, to appear Part II]
BPS States

BPS states of 5d gauge theories arise as M2-branes wrapped on curves $C$. For rational curves, i.e. $g(C) = 0$, the BPS states transforms under the 5d massive little group $SO(4)$ as

$$R_n = \left(\frac{n}{2}, \frac{1}{2}\right) \oplus 2 \left(\frac{n}{2}, 0\right),$$

$n=$dimension of the moduli space $\mathcal{M}_C$ [Gopakumar, Vafa].

Here: $C$= non-negative linear combination of curves in the CFD.

For $n = 0$ ‘spin 0’ states we find e.g. for the $E_n$ theories (rank 1): agreeing with [Huang, Klemm,Poretschkin]
<table>
<thead>
<tr>
<th>CFD for SCFT</th>
<th>SCFT Flavor</th>
<th>Gauge Theory</th>
<th>BPS Spin 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Diagram]</td>
<td>$E_8$</td>
<td>$SU(2) + 7F$</td>
<td>248</td>
</tr>
<tr>
<td>[Diagram]</td>
<td>$E_7$</td>
<td>$SU(2) + 6F$</td>
<td>56</td>
</tr>
<tr>
<td>[Diagram]</td>
<td>$E_6$</td>
<td>$SU(2) + 5F$</td>
<td>27</td>
</tr>
<tr>
<td>[Diagram]</td>
<td>$SO(10)$</td>
<td>$SU(2) + 4F$</td>
<td>16</td>
</tr>
<tr>
<td>[Diagram]</td>
<td>$SU(5)$</td>
<td>$SU(2) + 3F$</td>
<td>10</td>
</tr>
<tr>
<td>[Diagram]</td>
<td>$SU(3) \times SU(2)$</td>
<td>$SU(2) + 2F$</td>
<td>$(3, 2)$</td>
</tr>
<tr>
<td>[Diagram]</td>
<td>$SU(2) \times U(1)$</td>
<td>$SU(2) + 1F$</td>
<td>$1_{-1}, 2_1$</td>
</tr>
<tr>
<td>[Diagram]</td>
<td>$SU(2)$</td>
<td>$SU(2)_0$</td>
<td></td>
</tr>
<tr>
<td>[Diagram]</td>
<td>$U(1)$</td>
<td>$SU(2)_\pi$</td>
<td>1</td>
</tr>
<tr>
<td>[Diagram]</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
Higher Rank? Sure, all we need is the Marginal CFD

\[ D_k - D_k \] minimal Conformal Matter. Marginal CFD is

\[
\begin{array}{c}
\text{\footnotesize -$1$} \\
\end{array}
\begin{array}{c}
\cdots \\
\end{array}
\begin{array}{c}
\text{\footnotesize -$1$} \\
\end{array}
\]

Weakly coupled gauge theory descriptions of marginal theory:

- \( SU(k - 2)_0 \) with \( 2kF \)
- \( 4F - SU(2) - \ldots - SU(2) - 4F \), with \( k - 5 \) \( SU(2) \)s nodes and theta angle 0
- \( Sp(k - 3) \) with \( 2kF \).

To determine the daughter CFDs, run algorithm:
\[ D_k - D_k \text{ cont'd.} \]

\# \((k - 2)^2 - 3\) descendant SCFTs,
\# \(2k - 6\) without known gauge theory description.
\# Flavor enhancement, e.g. for models with \(SU(k - 2)_{\kappa} + mF\) gauge descr.

\[
\kappa \quad \text{SCFT Flavor Symmetry } G_F
\]

\[
k - \frac{m}{2} : \begin{cases} \quad SO(4k) \quad m = 2k - 1 \\ \quad SO(4k - 4) \times SU(2) \quad m = 2k - 2 \\ \quad SO(2m) \times U(1) \quad m = 0, \ldots, 2k - 3 \end{cases}
\]

\[
k - 1 - \frac{m}{2} : \begin{cases} \quad SU(2k) \quad m = 2k - 2 \\ \quad SU(2k - 2) \times SU(2) \quad m = 2k - 3 \\ \quad SU(m + 1) \times U(1) \quad m = 0, \ldots, 2k - 4 \end{cases}
\]

\[
k - 2 - \frac{m}{2} : \begin{cases} \quad SU(2k - 4) \times SU(2)^2 \quad m = 2k - 4 \\ \quad U(m) \times SU(2) \quad m = 0, \ldots, 2k - 5 \end{cases}
\]

In agreement with recent results in [Cabrera, Hanany, Zajac, 10/2018].
Higher Rank, cont’d

\((E_6, E_6)\) minimal Conformal Matter (CM): marginal theory has CFD

We find 207 descendant CFDs/SCFTs, including flavor symmetry and tree structure. Only known [del Zotto, Heckman, Tomasiello, Vafa] weakly coupled quiver description

\[
\begin{array}{c}
[2] \\
\downarrow \\
SU(2) \\
\downarrow \\
[2] - SU(2) - SU(3)_0 - SU(2) - [2].
\end{array}
\]

We find additional 195 theories that have no known gauge theory description. Similar structure for other CM matter theories.
.... because they add flavor to 5d SCFTs!
Summary and Outlook

• Given a 6d SCFT, we provide a systematic exploration of all descendant 5d SCFTs.

• Each 5d SCFT is characterized in terms of a CFD graph, which manifestly encodes the superconformal flavor symmetry, and spin 0 BPS states.

• Provides classification of 5d SCFTs under assumption that they all descend from 6d ones.
“Geometry and Strings 2019”
(aka Geometry and Physics of F-theory)

https://sites.google.com/view/geometryandstrings2019/home