Anomalies in the Space of Coupling Constants
Nathan Seiberg
IAS
C. Córdova, D. Freed, H.T. Lam, NS
Introduction

• It is standard to explore a classical or quantum theory as a function of its parameters. Two complementary applications:
  – Family of theories (e.g. Berry phase)
  – Make the coupling constants spacetime dependent
• More generally, we should view every theory as a function of
  – Spacetime dependent coupling constants
  – Spacetime dependent background gauge fields for global symmetries
  – Spacetime geometry (and associated choices like spin-structure, etc.)
  – Explore defects (e.g. interfaces) as these vary in spacetime
Introduction

We will discuss “generalized anomalies” in this large space of backgrounds and will examine two classes of applications

• An anomaly in a (multi-parameter) family of theories can lead, à la ‘t Hooft, to constraints on its long-distance behavior, e.g. predict phase transitions.
  – This is also useful as a test of dualities

• The coupling constants can vary in spacetime leading to a defect. The anomaly constrains the dynamics on the defect.

Since this view unifies many different, recently-studied phenomena, some of the results may seem familiar to many of you.
Introduction

This view will streamline, unify, and strengthen many known results, and will lead to new ones.

• We will start by demonstrating them in very simple examples (QM of a particle on a ring, 2d $U(1)$ gauge theory).
  – Since these examples are elementary, the more powerful formalism is not essential. But the examples provide a good pedagogical way to demonstrate the formalism.
• Then we will briefly turn to more advanced and more recently studied examples.
• We’ll end by revisiting the free 4d fermion.
Quantum mechanics of a particle on a ring

\[ \mathcal{L} = \frac{1}{2} \dot{q}^2 + \frac{i}{2\pi} \theta \dot{q} \quad \text{with} \quad q \sim q + 2\pi \]

\[ \theta \sim \theta + 2\pi \]

Spectrum \[ E_n = \frac{1}{2} \left( n - \frac{\theta}{2\pi} \right)^2 \]

Symmetries:

- \( U(1) \): \( q \rightarrow q + \alpha \)
- \( T \): \( q(\tau) \rightarrow -q(-\tau) \)
- \( \theta \in \pi \mathbb{Z} \), also \( C \): \( q \rightarrow -q \) (and therefore also \( CT \) ) combining to \( O(2) \)
- \( \theta \in \pi(2 \mathbb{Z} + 1) \), the Hilbert space is in a projective representation.

- A mixed \( C-U(1) \) anomaly [Gaiotto, Kapustin, Komargodski, NS]. This guarantees level-crossing there.
Quantum mechanics of a particle on a ring

[Gaiotto, Kapustin, Komargodski, NS]

Couple to a background $U(1)$ gauge field $A$

$$\mathcal{L} = \frac{1}{2} (\dot{q} + A)^2 + \frac{i}{2\pi} \theta (\dot{q} + A) + i \ k A$$

$k$ is a coupling for the background field only – a counterterm.

With nonzero $A$ the $\theta$ periodicity is modified

$$(\theta, k) \sim (\theta + 2\pi, k - 1)$$

- Related to that, the spectrum $E_n = \frac{1}{2} \left( n - \frac{\theta}{2\pi} \right)^2$ is invariant under $\theta \rightarrow \theta + 2\pi$, but the states are rearranged $|n\rangle \rightarrow |n + 1\rangle$. 
Quantum mechanics of a particle on a ring

[Gaiotto, Kapustin, Komargodski, NS]

\[ \mathcal{L} = \frac{1}{2} \dot{q}^2 + \frac{i}{2\pi} \theta \dot{q} \]

Break \( U(1) \to \mathbb{Z}_N \) with a potential, e.g. \( V(q) = \cos(Nq) \).

The qualitative conclusions are unchanged.

• For \( N \) even, we still have a similar \( \mathcal{C} - \mathbb{Z}_N \) anomaly at \( \theta = \pi \) and hence the ground state is degenerate there.

• For \( N \) odd, there is no anomaly – not a projective representation. But there is still degeneracy at \( \theta = \pi \).

  – In terms of background fields, we can preserve \( \mathcal{C} \) at \( \theta = \pi \) by adding a counterterm \( i \frac{N-1}{2} A \) for the \( \mathbb{Z}_N \) gauge field \( A \).

But then there is no \( \mathcal{C} \) at \( \theta = 0 \) (referred to as “global inconsistency”).
Quantum mechanics of a particle on a ring

\[ \mathcal{L} = \frac{1}{2} \dot{q}^2 + V(q) + \frac{i}{2\pi} \theta \dot{q} \]

Break \( \mathcal{C} \) and \( \mathcal{T} \) for all \( \theta \), e.g. by making \( V(q) \) generic (but \( \mathbb{Z}_N \) invariant) and adding degrees of freedom.

Now, the symmetry is only \( \mathbb{Z}_N \) (no \( U(1) \) and no \( \mathcal{C} \)).

Add a \( \mathbb{Z}_N \) background field \( A \) and a counterterm \( ikA \) (with \( k \) an integer modulo \( N \)). Again

\[ (\theta, k) \sim (\theta + 2\pi, k - 1) \]

Continuously shifting \( \theta \) by \( 2\pi \), the states are rearranged – a state with \( \mathbb{Z}_N \) charge \( l \) is mapped to a state with a \( \mathbb{Z}_N \) charge \( (l + 1)\mod N \).

The ground state must jump (level-crossing) at least once in \( \theta \in [0, 2\pi) \). There is a “phase transition.”
Quantum mechanics of a particle on a ring

\[ \mathcal{L} = \frac{1}{2} \dot{q}^2 + V(q) + \frac{i}{2\pi} \theta \dot{q} \]

\( (\theta, k) \sim (\theta + 2\pi, k - 1) \)

Two views on the parameter space

- Either \( \theta \sim \theta + 2\pi N \) and preserve the \( \mathbb{Z}_N \) symmetry
- Or \( \theta \sim \theta + 2\pi \), but violate the \( \mathbb{Z}_N \) symmetry

Generalized anomaly between the \( \mathbb{Z}_N \) symmetry and \( \theta \sim \theta + 2\pi \) (related discussion in [Thorngren; NS, Tachikawa, Yonekura]).

The anomaly is characterized by the 2d bulk term \( \frac{i}{2\pi} \int d\theta A \).
(Below it will be defined carefully.)
Quantum mechanics of a particle on a ring

Anomaly between the global $\mathbb{Z}_N$ symmetry and the $\theta$ periodicity.

It is characterized by the 2d bulk term $\frac{i}{2\pi} \int \theta dA$

• Viewing our system as a one-parameter family of theories labeled by $\theta$ the anomaly means that there must be level-crossing – “a phase transition.” (Alternatively, this follows from tracking the $\mathbb{Z}_N$ charge of the ground state.)

• “Interfaces” where $\theta$ changes as a function of time carry $\mathbb{Z}_N$ charge.

• In order to discuss the interfaces we need to define the action more carefully.
Quantum mechanics of a particle on a ring

When $\theta \in S^1$ is time dependent $\oint \theta(\tau)\dot{q}(\tau) d\tau$ should be defined more carefully (differential cohomology).

Divide the Euclidean-time circle into patches $\mathcal{U}_I = (\tau_I, \tau_{I+1})$, where $\theta$ is a continuous function to $\mathbb{R}$ (no $2\pi$ jump) and it jumps by $2\pi m_I$ with $m_I \in \mathbb{Z}$ on overlaps of patches.

$$\frac{1}{2\pi} \oint \theta(\tau)\dot{q}(\tau) d\tau \equiv \left( \sum_I \frac{1}{2\pi} \int_{\mathcal{U}_I} \theta dq + m_I q(\tau_I) \right) \mod 2\pi$$

- Independent of the trivialization
- Invariant under $q \to q + 2\pi$ and under $\theta \to \theta + 2\pi$
- If $\theta(\tau)$ has nonzero winding $M = \sum_I m_I$, this term is not invariant under $U(1): q \to q + \alpha$.

This is our anomaly.
3 kinds of “interfaces”

Smooth, universal
\[ \theta = 0 \]
\[ \theta = 2\pi m \]

Discontinuous, not universal
(can add an operator)
\[ \theta = 0 \]
\[ \theta = 2\pi m \]

Discontinuous with \( e^{imq} \).
It is transparent
\[ \theta = 0 \]
\[ \theta = 2\pi m \]
This is a rapid change in the Hamiltonian at some Euclidean time. In the “sudden approximation” it relabels the states $|n\rangle \rightarrow |n + m\rangle$, which can be achieved by multiplying by $e^{imq}$. This means that the interface has $U(1)$ charge $m$. When $\theta$ winds $m$ times around the Euclidean circle, the $U(1)$ symmetry is violated by $m$ unites.
This anomaly is related to a “symmetry”

Normally an anomaly is associated with a global symmetry: 0-form, 1-form, etc.

We can think of this anomaly as associated with a “-1-form symmetry.”

Just as -1-branes (instantons) are not branes, a -1-form global symmetry is not a symmetry.

If we view it as a global symmetry,

• $\theta$ is its gauge field (its transition functions involve jumps by $2\pi \mathbb{Z}$)
• $d\theta$ is the field strength (well defined even across overlaps)
• Then, this anomaly follows the standard anomaly picture.
2d $U(1)$ gauge theory

Consider a 2d $U(1)$ gauge theory with $N$ charge-one scalars $\phi^i$ and impose that the potential $V(|\phi|^2)$ is $SU(N)$ invariant.

Include $\frac{i}{2\pi} \theta da$.

We are not going to assume charge conjugation symmetry.

- The system has a $PSU(N)$ global symmetry and we couple it to a background $PSU(N)$ gauge field $B$ with the counterterm

$$2\pi i \frac{k}{N} w_2(B)$$

$w_2(B)$ is the characteristic class that measures the obstruction to lifting the $PSU(N)$ bundle to $SU(N)$. $k$ is an integer modulo $N$.

- Now $(\theta, k) \sim (\theta + 2\pi, k - 1)$ [Gaiotto, Kapustin, Komargodski, NS].
2d $U(1)$ gauge theory

• Now $(\theta, k) \sim (\theta + 2\pi, k - 1)$ [Gaiotto, Kapustin, Komargodski, NS].

• This can be viewed as a mixed anomaly between the periodicity of $\theta$ and the global $PSU(N)$ symmetry.

• It is described by the 3d anomaly term (see also [Thorngren])

$$\frac{i}{N} \int d\theta w_2(B)$$
2d $U(1)$ gauge theory

$$\frac{i}{N} \int d\theta w_2(B)$$

Use the anomaly as in the QM example.

- A one-parameter family of theories: since the system is gapped, it must have a phase transition at some $\theta$.
  - Note that we do not assume charge conjugation symmetry.

- A smooth interface, where $\theta \rightarrow \theta + 2\pi m$ must have an anomalous QM system on it – a projective $PSU(N)$ representation with $N$-ality $m$.
  - Can interpret as $m$ charged particles due to Coleman’s mechanism.
4d SU(N) gauge theory

This system has a one-form $\mathbb{Z}_N$ global symmetry. Couple it to a classical two-form $\mathbb{Z}_N$ gauge field $B$. It twists the $SU(N)$ bundle to a $PSU(N)$ bundle with [Kapustin, NS]

$$w_2(a) = B,$$

where $w_2(a)$ is the obstruction to lifting the $PSU(N)$ bundle to $SU(N)$ – ’t Hooft twisted configurations.

Mixed anomaly between the $\mathbb{Z}_N$ one-form symmetry and $\theta \sim \theta + 2\pi$ characterized by the 5d term

$$i \frac{N - 1}{2N} \int d\theta \mathcal{P}(B)$$

with $\mathcal{P}(B)$ the Pontryagin square of $B$. 
4d $SU(N)$ gauge theory

\[ i \frac{N - 1}{2N} \int d\theta \, \mathcal{P}(B) \]

This leads to earlier derived results:

- In $\mathcal{N} = 1$ SUSY, a mixed anomaly between the $\mathbb{Z}_{2N}$ R-symmetry and the $\mathbb{Z}_N$ one-form symmetry. Hence, a TQFT on domain walls [Gaiotto, Kapustin, NS, Willett] in agreement with the string construction of [Acharya, Vafa].

- …
4d $SU(N)$ gauge theory

\[ i \frac{N - 1}{2N} \int d\theta \, \mathcal{P}(B) \]

- Without SUSY, a mixed anomaly between $\mathcal{T}$ (equivalently, $\mathcal{CP}$) and the $\mathbb{Z}_N$ one-form symmetry at $\theta = \pi$ (for even $N$) implies a transition at $\theta = \pi$ (or elsewhere) [Gaiotto, Kapustin, Komargodski, NS]
  - Can use the new anomaly in similar systems without $\mathcal{T}$, (e.g. add a massive scalar with coupling $i\phi \, Tr(F \wedge F)$). The gapped system must have a transition for some $\theta$.
- Nontrivial TQFT on interfaces where $\theta$ changes by $2\pi m$ [Gaiotto, Komargodski, NS; Hsin, Lam, NS]
3 kinds of interfaces

- Smooth, universal
  \[ \theta = 0 \]
  \[ \theta = 2\pi m \]

- Discontinuous, not universal
  (can add d.o.f on the interface)
  \[ \theta = 0 \]
  \[ \theta = 2\pi m \]

- Discontinuous, with special QFT
  (e.g. a CS term). It is transparent
  \[ \theta = 0 \]
  \[ \theta = 2\pi m \]
A free massive Weyl fermion in 4d

• The parameters: a complex mass $m$.

• Upon compactification of the $m$ plane, a nontrivial 2-cycle in the parameter space.

• The phase of $m$ can be removed by a chiral rotation $\psi \rightarrow e^{i\alpha} \psi$. But because of a $U(1)$-gravitational anomaly, the Lagrangian is shifted by $\frac{i\alpha}{192\pi^2} Tr(R \wedge R)$.

• As above, we should add the counterterm $\frac{i \theta_G}{384\pi^2} Tr(R \wedge R)$, and then we should identify $(m, \theta_G) \sim (e^{i\alpha} m, \theta_G + \alpha)$.
A free massive Weyl fermion in 4d

\[(m, \theta_G) \sim (e^{i\alpha} m, \theta_G + \alpha)\]

- For nonzero \(m\) we can remove the anomaly, i.e. absorb the phase of \(m\), in a redefinition: \(\theta_G \rightarrow \theta_G - \text{arg}(m)\). But we cannot do it for all complex \(m\).

- This can be described as an anomaly using the 5d term

\[i \int \delta^{(2)}(m) d^2m \, CS_g\]

\(CS_g\) is the gravitational Chern-Simons term.

Equivalently, this can be written as the 6d term

\[\frac{i}{192\pi} \int \delta^{(2)}(m) d^2m \, Tr(R \wedge R)\]

- Related discussion in [NS, Tachikawa, Yonekura].
A free massive Weyl fermion in 4d

\[ \frac{1}{192\pi} \delta^{(2)}(m) d^2 m \text{Tr}(R \wedge R) \]

Applications:

• A 2-parameter family of theories. At some point in the complex \( m \) plane the theory is not trivially gapped.
  – For the free fermion this is trivial. But the same conclusion in more complicated models, with additional fields and interactions. Note, we do not need any global symmetry.

• Defects. \( m \sim (x + iy) \) is a string along the \( z \) axis. As in \cite{Jackiw, Rossi}, the anomaly means that there are Majorana-Weyl fermions \( (c_L - c_R = 1/2) \) on the defect.
  – The same conclusion in a more complicated theory with more fields and interactions.
Conclusions

• To study ordinary anomalies we couple every global symmetry to a classical background field and place the theory in an arbitrary spacetime. Denote all these background fields by $A$. An anomaly is the statement that the partition function might not be gauge invariant or coordinate invariant.

• This is described by a higher dimensional classical (invertible) field theory for $A$.

• Similarly, we make all the coupling constants $\lambda$ background fields and then the partition function might not be invariant under identifications in the space of coupling constants.

• This is described by a generalized anomaly theory – a higher dimensional classical (invertible) field theory for $A$ and $\lambda$. 
Conclusions

• This anomaly theory is invariant under renormalization group flow. (More precisely, it can be deformed continuously). Therefore, it can be used, à la ‘t Hooft, to constrain the long-distance dynamics and to test dualities.
  – For example, it can force the low-energy theory to have some phase transitions.
• Defects are constructed by making $A$ and $\lambda$ spacetime dependent. Using these values in the anomaly theory, we find the anomaly of the theory along the defect.
Conclusions

• We presented examples of these anomalies (and their consequences) in various number of dimensions. The parameter space in the examples has one-cycles or two-cycles.
• We have studied many other cases with related phenomena.
  – 4d $SU(N), Spin(N), Sp(N)$ with quarks
• There are many other cases, we have not yet studied.