Elliptic and genus one fibration structure of known Calabi-Yau threefolds

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Based in part on

arXiv: 1805.05907, 1809.05160, 1811.04947, 1907.xxxxx, Y-C. Huang, WT

and upcoming work in collaboration with various subsets of:
L. Anderson, P. Berglund J. Gray, J. Halverson, Y. Huang, C. Long, P. Oehlmann

Outline

1. Calabi-Yau threefolds and elliptic fibers

2. Toric polytope fibrations

3. Triangulations

4. Mirror symmetry and elliptic fibrations

Calabi-Yau threefolds: manifolds used in superstring compactification

physically:

- Ricci flat: $R_{\mu\nu} = 0$ (solve vacuum Einstein equations)
- Kähler manifolds (complex structure compatible with SUSY)

mathematically: trivial canonical class $K = 0$ (up to torsion)

Long studied by mathematicians and physicists

— Used in compactification of string theory: 10D $\rightarrow$ 4D, 6D, ...

Open Question:

Are there a finite number of topological types of Calabi-Yau threefolds?

Many large classes of CY3s have been constructed:

— CICY (complete intersection CYs): 7,890 [Candelas/Dale/Lutken/Schimmrigk]
— Toric hypersurface CY3s: 473.8M reflexive 4D polytopes [Kreuzer/Skarke]
— Generalized CICYs [Anderson/Apruzzi/Gao/Gray/Lee]
— Elliptic CY3s (this talk)

Elliptic and genus one-fibered CY threefolds

An *elliptic* or *genus one fibered* CY3 X:
 A torus (fiber) at each $p \in B_2$
 $\pi : X \to B_2$

Elliptic: $\exists$ section $\sigma : B_2 \to X, \pi\sigma = \mathrm{Id}$

- Elliptic Calabi-Yau threefold has Weierstrass model

$$y^2 = x^3 + fx + g, \quad f, g \text{ sections of } \mathcal{O}(-4K_B), \mathcal{O}(-6K_B)$$

- Elliptic CY3s have extra structure, more manageable mathematically
- Used for 6D F-theory construction (fiber $\tau =$ 10D axiodilaton)
- Recently evidence has been accumulating that
 most known Calabi-Yau threefolds have elliptic/g1 structure
 (i.e. birationally equivalent to an elliptic or genus one fibered CY3)
- F-theory + math $\Rightarrow$ global picture of { ECY3s }

Classifying elliptic Calabi-Yau threefolds

$\exists$ finite # of topological types of elliptic CY3s [Gross]

Combining math + physics (F-theory), can systematically construct ECY3s

- Bases B_2: must be $\mathbb{F}_m, \mathbb{P}^2$, Enriques or blow-ups [Grassi]
- Finite number of distinct strata in space of B_2 Weierstrass models

"Algorithm" for constructing all ECY3s:

I. Construct bases by iterative blow-ups of $\mathbb{F}_m, \mathbb{P}^2$ [large $h^{2,1}$ ✓]

II. Tune Weierstrass models

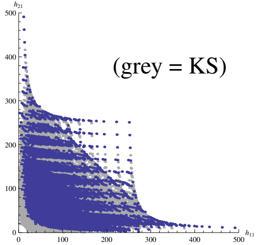
A. Codimension 1 singularities $\leftrightarrow$ nonabelian G (Kodaira) [mostly ✓]

B. Mordell-Weil rank $\leftrightarrow U(1)^k$ [$k \leq 2$ ✓]

C. Discrete G ($\sim$ multisections) [no systematics yet X]

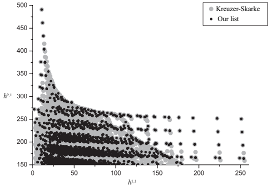
D. Codim. 2 singularities $\leftrightarrow$ matter [generic NA ✓, exotic, $U(1) \times G'$?]

Systematic construction of bases



Generic EF on 61,539 toric bases

[Morrison/WT]



Generic EF on non-toric bases $h^{2,1} \geq 150$

[WT/Wang]

Can systematically classify bases,
bounded by non-Kodaira singularities (e.g. $C \cdot C < -12$) not allowed

New Hodge numbers for generic EF over non-toric bases at
$(h^{1,1}, h^{2,1}) = (29, 299), (48, 270), \ldots$

Toric hypersurface construction [Batyrev, Kreuzer/Skarke]

Toric variety: characterized by toric divisors $D_i \leftrightarrow$ rays $v_i \in \mathbb{Z}^d$

Anti-canonical class $-K = \sum_i D_i$ (never compact CY)

Anti-canonical hypersurface $\Rightarrow$ CY by adjunction

∇ polytope: convex hull of vertices v_i

$\{$monomials$\} \leftrightarrow$ lattice points in *dual polytope* $\nabla^* = \{w : w \cdot v \geq -1\}$

Batyrev: $\nabla = \nabla^{**}$ reflexive $\leftrightarrow$ 1 interior point
$\leftrightarrow$ hypersurface CY generically smooth (avoids singularities)

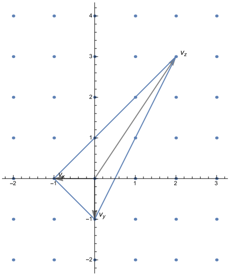
Kreuzer-Skarke: constructed all 473.8M reflexive ∇_4

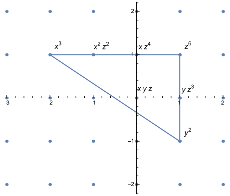
∇, Δ describe mirror Calabi-Yau threefolds $h^{1,1} \leftrightarrow h^{2,1}$

Symmetry in toric hypersurface construction early evidence for mirror symmetry

Example: Batyrev for generic elliptic curve in $P^{2,3,1}$

∇





Δ

Gives general Weierstrass ("Tate form") model for elliptic curve:

$$y^2 + a_1 yx + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

Completing square, cube $\rightarrow$ standard (short) Weierstrass form

$$y^2 = x^3 + fx + g$$

Why likely a CY3 has an elliptic or genus one fiber?

Theorem (Oguiso/Wilson):
A Calabi-Yau 3-fold X, X is genus one (or elliptically) fibered iff there exists a divisor $D \in H^2(X, \mathbb{Q})$ that satisfies $D^3 = 0, D^2 \neq 0$, and $D \cdot C \geq 0$ for all algebraic curves $C \subset X$.

Assuming "random" data for triple intersection form C_{ijk},
how likely is this to occur?

Possible obstructions:

A) Number theoretic (no solution to $C_{ijk} d_i d_j d_k = 0$ over integers)

B) Cone obstruction, no solution over reals when $D \subset$ positive cone

Consider each in turn

Number theoretic obstructions

For example:

$$x^3 + x^2y + y^3 + 2z^3 + 4w^3 = 0$$

has no solutions over the integers $\mathbb{Z}$ (or over $\mathbb{Q}$); ($\mathbb{Z}_2$ obstruction)

Mordell (1937) identified homogeneous degree d polynomial in d^2 variables with obstruction

Subsequent conjectures: d^2 is maximum number of variables with obstruction

Proven for $d = 1, 2$

Counterexample: quartic with 17 variables has obstruction!

Heath-Brown (1983): every non-singular cubic in ≥ 10 variables with rational coefficients has nontrivial rational zero.

Also proven for general cubic in ≥ 16 variables

Upshot: no number-theoretic obstruction when $h^{1,1}(X) > 15$ (likely 9)

Cone obstructions: exponentially suppressed?

Simple heuristic argument:

Assume cone has $D = \sum_i d_i D_i, d_i \geq 0$

Look for positive solution of cubic $\sum_{i,j,k} C_{ijk} d_i d_j d_k = 0$

Proceed by induction:

First, check $M = 2$, $\sum_{i,j,k}^{M} C_{ijk} d_i d_j d_k = 0$
$\sim$ cubic in two variables, has ≥ 1 real solution; 50% chance in cone

Add one variable: pick random other numbers in cone; probability solution in last variable is positive: 1/2, ...

$\Rightarrow$ suggests probability $\leq\sim 2^{-h^{1,1}}$ that no fiber exists

Very heuristic argument, depends on "large" Kaehler cone

Finding elliptic fibrations in known CY3's

CICYs analyzed by Anderson/Gao/Gray/Lee:

- 99.3% (7837/7890) of CICYs have "obvious" elliptic/g1 fibration
- Average number of inequivalent obvious fibrations: 9.85
- At least one fibration whenever $h^{1,1} \geq 4$

How about toric hypersurface CY3s

With Y.-C. Huang have taken two systematic approaches:

A) Construct elliptic CY3s by choosing toric bases, tuning, and compare to KS

B) Directly studying toric hypersurfaces and reflexive polytopes for fibrations

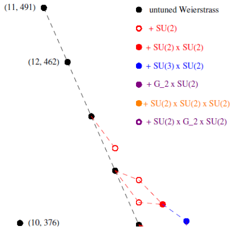
Next describe results of these two approaches

Constructing elliptic CY3s and comparing to KS

Start with toric base, Tate tuning $\sim$ "tops" of polytopes

Can systematically construct tunings that should correspond to polytopes

Example: $(11, 491)$ = generic EF over $\mathbb{F}_{12}$
$(12, 462)$ = generic EF over $\mathbb{F}_{12}$
blown up at generic pt.
$(13, 433)$: can tune SU(2) on exceptional curve
$\rightarrow (14, 416)$ [Johnson/WT]



w/Huang: Systematically tuned models at $(h^{1,1}$ or $h^{2,1} \geq 240)$

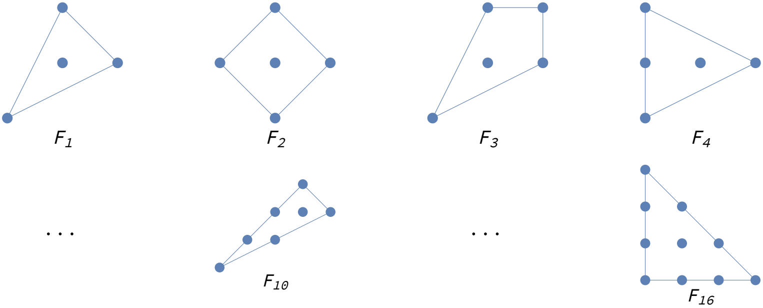
- Initial sieve gave all Hodge numbers in KS list except 18
- Checked remaining cases explicitly, all 18 elliptic fibered (some large tunings, non-flat fibrations, forced Mordell-Weil)
- Identified new SU(6) Tate tuning (with 3-index antisymmetric matter)

Upshot: tuning W. models over toric bases reproduces all large KS Hodge #'s

Simple toric fibrations:

$\nabla_2 \subset \nabla$, ∇_2 reflexive

Only 16 reflexive ∇_2's (e.g. F-theory fibers:
[Braun, Braun/Grimm/Keitel, Klevers/Mayorga Pena/Oehlmann/Piragua/Reuter])



-1 curve $C = D_i^{(2)}$: satisfies $-K \cdot C = C \cdot C + 2 = 1$

All but $F_1 = \mathbb{P}^2, F_2 = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1, F_4 = \mathbb{F}_2$ have -1 curves $\Rightarrow$ toric sections

Check explicitly for fibrations: scan all reflexive ∇ for $\nabla_2 \subset \nabla$

For efficiency: start with $L \in \nabla$, V vertices of Δ

$$\text{Define } S_n = \{p \in L : \max_{v \in V} \langle p, v \rangle = n\}$$

Necessary & sufficient condition for fiber $\nabla_2 \subset \nabla$,
one of these conditions hold:

I $\exists x, y \in S_1 : x \neq y$ and $-x, -y \in S_1$

II $\exists x, y \in S_2 : -(x+y) \in S_1$ or $-(x+y) \in S_2$

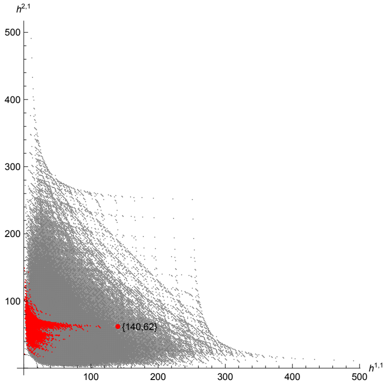
III $\exists x, y \in S_3 : -(x+y)/2 \in S_1$.

Checked all 473.8 M polytopes

$h^{1,1}$ or $h^{2,1} \geq 140$: Mathematica code [arXiv:1809.05160]
All ∇: Julia code (results to appear)

Toric fibers

Upshot: all but 29,223 4D reflexive polytopes have 2D fiber $\nabla_2 \subset \nabla$



Largest $h^{2,1}$ without toric fiber: $(140, 62)$

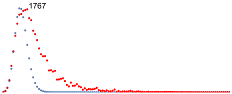
Those without toric fiber: $h^{2,1} \sim 60$ as $h^{1,1} \to 100$. Why?

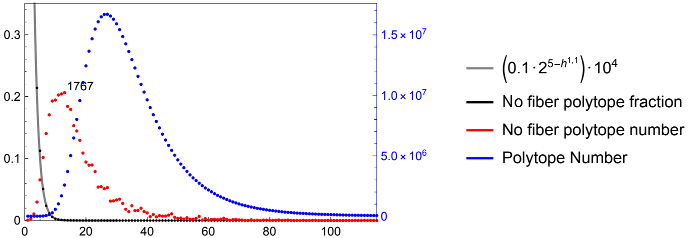
Asymptotics at small $h^{1,1}$:

Probability that a CY3 is not g1/elliptic fibered $\sim 2^{-h^{1,1}}$ for $1 < h^{1,1} \lesssim 10$

$h^{1,1}$	2	3	4	5	6	7
# without fiber ∇_2	23	91	256	562	872	1202
Total #	36	244	1197	4990	17101	50376
%	0.639	0.373	0.214	0.113	0.051	0.024

But there is a fat tail





Why? (Non-toric fibers? [$\sim$ *AGGL*]

Narrow Kähler cone? [Demirtas/Long/McAllister/Stillman]

Fibers and triangulations (work in progress)

Almost all KS polytopes have reflexive toric 2D subpolytopes

What does this mean about corresponding CY threefolds?

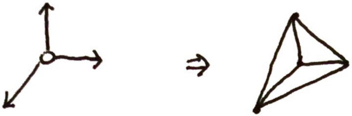
Heuristically, suggests elliptic/genus one fibrations

Consider triangulations (Fine, Regular, Star) $\to$ smooth CY3s

Is there always a compatible triangulation? [Rohsiepe]

For simple cases, all triangulations $\to$ toric morphisms (good E/G1 fiber)

Example: (2, 272); generic EF over $\mathbb{P}^2$ (fiber F_{10})

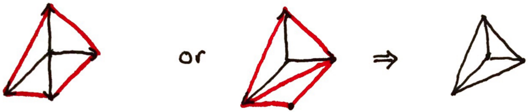


Single triangulation of ∇ and B_2, $\to$ toric morphism, elliptically fibered

Triangulations II

Many triangulations lead to non-flat fibrations [Braun]
(cf. also e.g. [Candelas/Diaconescu/Florea/Morrison/Rajesh, Lawrie/Schafer-Nameki, Dierigl/Oehlmann/Ruehle] re. non-flat fibrations in F-theory)

Example: (3, 243) generic EF over $\mathbb{F}_1$ (fiber F_{10})

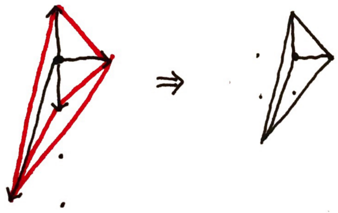


Two triangulations of $\nabla \Rightarrow 2$ CY3 phases
1) Good toric morphism, EF; 2) Good toric morphism, non-flat fiber

Triangulations III (work with Berglund)

In some cases FRS triangulation $\rightarrow$ singular base

(cf. e.g. [Anderson/Grassi/Gray/Oehlmann]) singular bases in F-theory



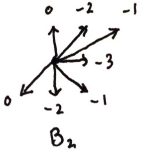
Given "simple stacking" polytope with base over $(-2,-3) \in \mathbb{P}^{2,3,1}$,
to find generic EF (fiber F_{10}) over $\mathbb{F}_3$ (w/ "non-Higgsable" A_2 singularity):

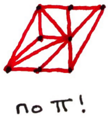
Need "vex" (non-convex) polytope [Berglund/Hubsch]

Not FRS triangulation but still CY3 phase connected by flops

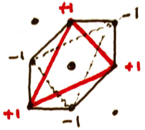
Triangulations IV

In some cases, an FRS triangulation is incompatible with any toric morphism
(work w/Halverson, Long)





For some polytopes there is no FRS triangulation compatible with toric morphism



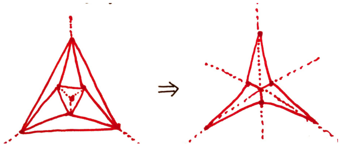
Triangulations V 

But always $\exists$ fine (non-star) triangulation compatible w/ toric morphism
$\rightarrow$ elliptic/genus one fibration in some phase
non-convex polytope: non-Fano variety with rigid surface $\subset$ CY3

Algorithm:

1. Take rays v_i that project to vertices of base (always primitive)
2. Use all rays in fiber to make 2D, 3D, 4D cones (singular, compatible)
3. Refine by adding remaining rays to get non-star triangulation

Examples:



$\rightarrow$ elliptic/genus one phase

Mirror symmetry

Mirror symmetry factorizes for many toric hypersurfaces!

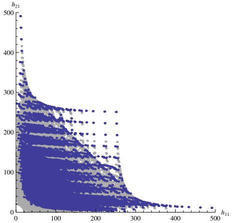
If $F = \nabla_2 \subset \nabla$ is a slice and $\tilde{F} = \Delta_2 \subset \Delta$ is also a slice
$\Rightarrow$ Mirror symmetry factorizes

Simplest factorization:
Standard stacking on $\mathbb{P}^{2,3,1} \leftrightarrow$ Tate form Weierstrass model

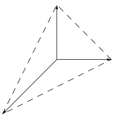
Mirror of generic elliptic fibration over B = ef over $\tilde{B}$ (may be tuned):

$$B \to \tilde{B} \sim \Sigma(-6K_B), \nabla_2 = \Delta_2 = \mathbb{P}^{2,3,1}$$

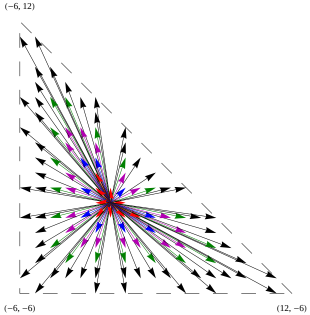
(65k examples in KS database)



Example: generic elliptic fibration on $\mathbb{P}^2$ (2, 272)



Hodge numbers (2, 272)



Hodge numbers (272, 2)
(toric rays: $\vec{w} \cdot \vec{v} \geq -6$, $\forall \vec{v} \in \Sigma_B$, $\vec{w}$ primitive)

$h^{1,1}(B) = 1$

$G = 1$

$h^{1,1}(X) = h^{1,1}(B) + \text{rk}\, G + 1 = 2$

$h^{2,1}(X) = 301 - 29h^{1,1}(B) - \dim M_{nh} = 272$

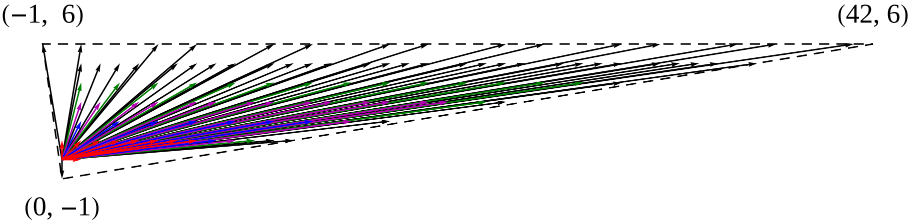
$h^{1,1}(B) = 106 + 3 = 109$

$G = E_8{}^9 \times F_4{}^9 \times (G_2 \times SU(2))^{18}$

$h^{1,1}(X) = h^{1,1}(B) + \text{rk}\, G + 1 = 272$

$h^{2,1}(X) = 301 - 29h^{1,1}(B) + \dim G - \dim M_{nh} = 2$

Example: self-mirror generic elliptic fibration (251, 251)



Toric self-intersections:

$$(0, 6, -12//-11//-12//-12//-12//-12//-12//-12//-12)$$
$$(// = (-1,-2,-2,-3,-1,-5,-1,-3,-2,-2,-1))$$

$h^{1,1}(B) = 97 + 1 = 98$

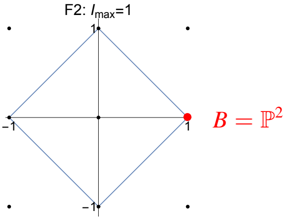
$G = E_8{}^9 \times F_4{}^8 \times (G_2 \times SU(2))^{16}$ (rank = 152)

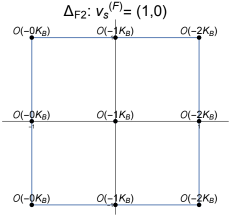
$h^{1,1}(X) = h^{1,1}(B) + \text{rk}\, G + 1 = 251$
$h^{2,1}(X) = 301 - 29h^{1,1}(B) + \dim G - \dim M_{nh} = 251$

Factorized mirror symmetry: more general structures

- Also works for "tuned" Tate models $\leftrightarrow$ reduction on Δ
- Works for other fibers, bundle structures

e.g. $B = \mathbb{P}^2, F = F_2$; base stacked over vertex: $H = (4, 94)$
$\tilde{B} \sim -2K_B, \tilde{F} = F_{15}$; $H = (94, 4)$
$(\tilde{B} : (-3, -1, -4, -1, -4, -1, -3, -1, -4, -1, -4, -1, -3, -1, -4, -1, -4, -1))$





(mirror symmetry of fibers:
discussed in [Klevers/Mayorga Pena/Oehlmann/Piragua/Reuter])

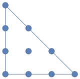
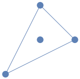
Mirror symmetry: weak Fano base + fiber (w/ Oehlmann)

Interesting class of cases:

Fiber F_i, base F_j both reflexive 2D polytopes (weak Fano/generalized dP)

$F_i \oplus F_j$ mirror to $\tilde{F}_i \otimes \tilde{F}_j$: $\sim$ Schoen (19, 19)

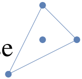
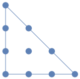
Example:

base  $\oplus$ 

F_{16} F_1

$(v; 0) + (0; v')$

$\leftrightarrow$

base  $\otimes$ 

F_1 F_{16}

$(\tilde{v}; \tilde{v}')$

$X : (8, 35)$; $G = \mathbb{Z}_3$ over $dP_6 \leftrightarrow \tilde{X} : (35, 8)$ $G = SU(9)^3$, w/ $9 \times (4, 6)$ points

Higgs $SU(9)^3 \to SU(8)^3 \times U(1) \to SU(3)^3 \times U(1)^6$: fiber $gdP_6 \to \mathbb{P}^2$
$(35, 8) \to (33, 9) \to \ldots$

Non-toric: Higgs 2 more times $\to$ Schoen (19, 19)

Factorize mirror symmetry for CY fourfolds

Same story for fourfolds:

$F = F_i \to \tilde{F} = F_{17-i}$ (except $\tilde{F} = F_i$ for $i = 7, 8, 9, 10$)
$B \to \tilde{B} \sim -nK_B$ for vertex stacking,

Example: $B = \mathbb{P}^3$ standard stacking ($F = \mathbb{P}^{2,3,1} = F_{10}$)

Rays in $\tilde{B}$: primitive lattice points in tetrahedron:
w/vertices (-6, -6, -6), (18, -6, -6), (-6, 18, -6), (-6, -6, 18)

$G = E_8^{34} \times F_4^{96} \times G_2^{256} \times SU(2)^{384}$

- (Exponentially) many triangulations
- Note: common endpoint from random blow-up sequence [WT/Wang]

Conclusions

- All but 29k of 483M toric hypersurface Calabi-Yau threefolds have elliptic/genus one structure (birational to elliptic/g1)
- In many cases mirror symmetry factorizes between fiber and base