Mirroring Towers of Feynman Integrals

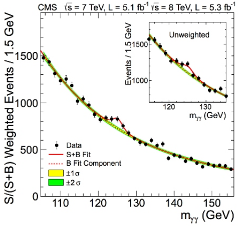
Pierre Vanhove

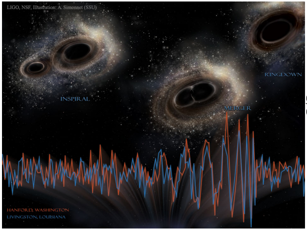
IPhT Saclay & HSE

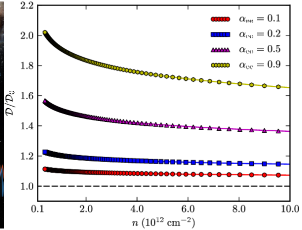
String Math 2019, Uppsala university, Uppsala

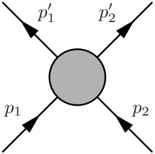
based on work to appear with

Charles Doran, Andrey Novoseltsev









Scattering amplitudes are the fundamental tools for making contact between quantum field theory description of nature and experiments

- Comparing particule physics model against datas from accelators
- Post-Minkowskian expansion for Gravitational wave physics
- Various condensed matter and statistical physics

Physics arguments indicate that any quantum field theory amplitude can be expanded on a **finite** basis of integral functions

$$A^{\mathrm{L-loop}}_{n-\mathrm{part.}} = \sum_{i \in \mathcal{B}(L)} \mathrm{coeff}_i \, \mathrm{Integral}_i + \mathrm{Rational\ function}$$

- What is the dimension of the basis $\mathcal{B}(L)$?
- What are the functions in the basis?
  - Feynman integrals are highly transcendental functions with a lot singularities
- How can we achieve such decomposition?

Feynman Integrals: parametric representation

The integral functions in the basis are Feynman integrals with L-loop and n internal edges

$$I_\Gamma(\underline{s},\underline{m}) = \Gamma\left(n - \frac{LD}{2}\right)\int_{\Delta_n}\Omega_\Gamma; \qquad \Omega_\Gamma := \frac{\mathcal{U}_\Gamma(\underline{x})^{n-\frac{(L+1)D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{n-\frac{LD}{2}}}\prod_{i=1}^{n-1} dx_i$$

The domain of integration is the positive quadrant

$$\Delta_n := \{x_1 \geqslant 0, \dots, x_n \geqslant 0 | [x_1, \dots, x_n] \in \mathbb{P}^{n-1}\}$$

If the integral diverges for $D \in \mathbb{N}^*$ then one considers the Laurent expansion

$$I_\Gamma(\underline{s},\underline{m}) = \sum_{r\geqslant -m} \epsilon^r I_\Gamma^{(r)}(\underline{s},\underline{m}) \qquad D = D_c - 2\epsilon; D_c \in \mathbb{N}^*; 0 \leqslant \epsilon \ll 1$$

Feynman Integrals: parametric representation

The graph polynomial is homogeneous degree $L+1$ in $\mathbb{P}^{n-1}$

$$\mathcal{F}_\Gamma(\underline{x}) = \mathcal{U}_\Gamma(\underline{x}) \times \left(\sum_{i=1}^{n} m_i^2 x_i\right) - \mathcal{V}_\Gamma(\underline{s}, \underline{x})$$

$$\mathcal{U}_\Gamma(\underline{x}) = \sum_{\substack{a_1+\cdots+a_n=L \\ 0\leqslant a_i\leqslant 1}} u_{a_1,\ldots,a_n} \prod_{i=1}^{n} x_i^{a_i}, \qquad \mathcal{V}_\Gamma(\underline{x}) = \sum_{\substack{a_1+\cdots+a_n=L+1 \\ 0\leqslant a_i\leqslant 1}} S_{a_i,\cdots,a_n} \prod_{i=1}^{n} x_i^{a_i}$$

$u_{a_1,\ldots,a_n} \in \{0,1\}$ and $S_{a_i,\cdots,a_n}$ are linear combination of the kinematic variables

From $\mathcal{F}_\Gamma$ we can reconstruct the associated Feynman graph Γ

- the number of edges is n
- the loop order is $L = \deg(\mathcal{F}_\Gamma) - 1$
- Number of vertices $v = 1 + n - L$ from Euler characteristic of the planar graph

Feynman Integrals: parametric representation

$$I_\Gamma(\underline{s},\underline{m}) = \Gamma\left(n - \frac{LD}{2}\right)\int_{\Delta_n}\Omega_\Gamma; \quad \Omega_\Gamma := \mathrm{Res}_{X_\Gamma}\left(\frac{\mathcal{U}_\Gamma(\underline{x})^{n-\frac{(L+1)D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{n-\frac{LD}{2}}}\prod_{i=1}^{n-1} dx_i\right)$$

Algebraic differential form $\Omega_\Gamma \in H^{n-1}(\mathbb{P}^{n-1}\backslash X_\Gamma)$ on the complement of the graph hypersurface

$$X_\Gamma := \{\mathcal{U}_\Gamma(\underline{x}) = 0 \& \mathcal{F}_\Gamma(\underline{x}) = 0, \underline{x} \in \mathbb{P}^{n-1}\}$$

- All the singularities of the Feynman integrals are located on the graph hypersurface
- Generically the graph hypersurface has non-isolated singularities

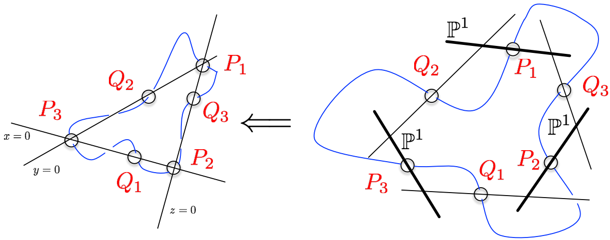
Feynman integral and periods

$\Delta_n \notin H^{n-1}(\mathbb{P}^{n-1}\backslash X_\Gamma)$ because

$$\partial\Delta_n \cap X_\Gamma = \{(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)\}$$

we have to look at the relative cohomology $H^\bullet(\mathbb{P}^{n-1}\backslash X_\Gamma; Д_n\backslash Д_n \cap X_\Gamma)$

The normal crossings divisor $Д_n := \{x_1\cdots x_n = 0\}$ and X_Γ are separated by performing a series of iterated blowups of the complement of the graph hypersurface [Bloch, Esnault, Kreimer]



Differential equation

The Feynman integral *are* periods of the relative cohomology after performing the appropriate blow-ups

$$\mathfrak{M}(\underline{s}, \underline{m}^2) := H^\bullet(\widetilde{\mathbb{P}^{n-1}}\backslash\widetilde{X_F}; \widetilde{\Pi_n}\backslash\widetilde{\Pi_n}\cap\widetilde{X_\Gamma})$$

Since Ω_Γ varies with the kinematic variables $\underline{s}$ and internal mass $\underline{m}$ one needs to study a variation of (mixed) Hodge structure

The Feynman integrals inhomogenous differential equation

$$L_{PF}\, I_\Gamma = S_\Gamma$$

Generically there is an inhomogeneous term $S_\Gamma \neq 0$ due to the boundary components $\partial\Delta_n$

Deriving this differential equation is difficult in general and requires a lot of computer resources and is still a major question in QFT

Gel’fand-Zelevinsky-Kapranov approach

Unitarity in QFT motivates considering the following period integral

$$\pi_\Gamma = \frac{1}{(2i\pi)^n} \int_{|x_1|=\cdots=|x_n|=\epsilon} \Omega_\Gamma \qquad 0 < \epsilon \ll 1$$

Consider the toric polynomial

$$\mathcal{F}^{\text{toric}}_{\Delta(\Gamma)}(\underline{x}) = \sum_{\nu \in \Delta(\Gamma)\cap\mathbb{Z}^{n+1}} f_\nu \, x^\nu$$

- $\mathcal{F}_\Gamma(\underline{x})$ is a specialisation of the toric deformation parameters to the physical locus $f_\nu \mapsto (\underline{s}, \underline{m})$. The map is *linear*
- π_Γ is a period integral for the Calabi-Yau hypersurface X_{Δ° and is a GZK $\mathcal{A}$-hypergeometric series

The Feynman graph hypersurface is highly non generic

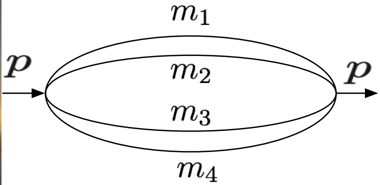
- The system often resonant and reducible
- Obtaining the minimal order Picard-Fuchs operator this way is not an easy task as one must restrict the D-module

Geometry for Feynman graph motives

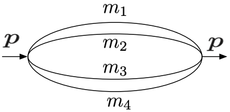
What is the geometry governing the Feynman graph motive?

1. One-loop graph hypersurface degree 2 in $\mathbb{P}^{n-1}$
   - The motive structure is the one of dilogarithms
2. Two-loop graph hypersurface degree 3 in $\mathbb{P}^{n-1}$
   - for $n = 3$ Brown-Levin elliptic multiple polylogarithms [Bloch, Vanhove; Adams, Bogner, Weinzierl]
   - For $n \geqslant 4$ motivic elliptic curve for the Hodge structure [Bloch, Doran, Kerr, Vanhove (work in progress)]
3. family of sunset graphs : $n-1$-loop graph hypersurface degree n in $\mathbb{P}^{n-1}$ define Calabi-Yau $n-2$-fold [Doran, Novoseltsev ,Vanhove] (this talk)

The sunset graphs family



The sunset family of graph



The graph polynomial for the $n-1$-loop sunset

$$\mathcal{F}_n^{\ominus}(\underline{x}) = x_1 \cdots x_n \left(\phi_n^{\ominus}(\underline{x}) - p^2\right);\ \phi_n^{\ominus}(\underline{x}) = \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)\left(m_1^2 x_1 + \cdots + m_n^2 x_n\right)$$

The Feynman integral in $D=2$

$$I_n^{\ominus}(p^2, \underline{m}^2) = \int_{x_1 \geqslant 0, \ldots, x_n \geqslant 0} \frac{1}{p^2 - \phi_n^{\ominus}(\underline{x})} \prod_{i=1}^{n-1} \frac{dx_i}{x_i}$$

and the classical period

$$\pi_n^{\ominus}(p^2, \underline{m}^2) = \int_{|x_1| = \cdots = |x_n| = 1} \frac{1}{p^2 - \phi_n^{\ominus}(\underline{x})} \prod_{i=1}^{n-1} \frac{dx_i}{x_i}$$

The sunset integrals and L-function values

For the special value $p^2 = m_1^2 = \cdots = m_n^2 = 1$ the sunset Feynman integral becomes a pure period integral [Bloch, Kerr, Vanhove]

$$I_n^{\ominus}(1,\ldots,1) = \int_{x_i \geqslant 0} \frac{\prod_{i=1}^{n-1} d\log x_i}{1 - \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)(x_1 + \cdots + x_n)}$$

Using impressive numeric experimentations [Broadhust] found that $I_n^{\ominus}(1,\ldots,1)$ is given by L-function values in the critical band. For large n the L-function are from moments Kloosterman sums over finite fields

The sunset integrals and L-function values

For the special value $p^2 = m_1^2 = \cdots = m_n^2 = 1$ the sunset Feynman integral becomes a pure period integral [Bloch, Kerr, Vanhove]

$$I_n^{\ominus}(1,\dots,1) = \int_{x_i\geqslant 0} \frac{\prod_{i=1}^{n-1} d\log x_i}{1-\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)(x_1+\cdots+x_n)}$$

These special values realise explicitly Deligne's conjecture relating period integrals to L-values in the critical band

$n=3$: elliptic curve case : $I_3^{\ominus}(1,\dots,1) = \frac{1}{2}\zeta(2)$
$n=4$: $K3$ Picard rank 19 : $I_4^{\ominus}(1,\dots,1) = \frac{12\pi}{\sqrt{15}} L(f_{K3},2)$ [Bloch, Kerr, Vanhove]

- $L(f_{K_3},s)$ is the L-function of $H^2(K3,\mathbb{Q}_\ell)$
- Functional equation $L(f_{K3},s) \propto L(f_{K3},3-s)$
- $f_{K3} = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)\sum_{m,n} q^{m^2+4n^2+mn}$

The sunset integrals and exponential motives

The Feynman integral for $0 \leqslant p^2 \leqslant (m_1 + \cdots + m_n)^2$

$$I_n^{\ominus}(p^2, \underline{m}^2) = 2^{n-1} \int_0^{\infty} u I_0(\sqrt{p^2} u) \prod_{i=1}^{n} K_0(m_i u) du$$

The classical period for $p^2 \geqslant (m_1 + \cdots + m_n)^2$

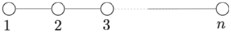
$$\pi_n^{\ominus}(p^2, \underline{m}^2) = \frac{1}{2} \int_0^{\infty} u K_0(\sqrt{p^2} u) \prod_{i=1}^{n} I_0(m_i u) du$$

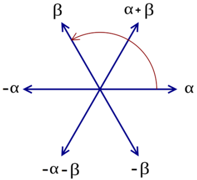
where we have the modified Bessel function of the first kind

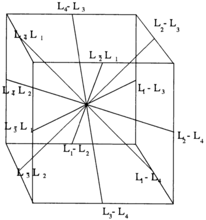
$$I_0(z) = \frac{1}{2i\pi} \int_{|t|=1} e^{-\frac{z}{2}\left(t+\frac{1}{t}\right)} d \log t; \qquad K_0(z) = \int_0^{+\infty} e^{-\frac{z}{2}\left(t+\frac{1}{t}\right)} d \log t$$

There are exponential period integrals in the sense of the non classical exponential motives developed by [Fresàn, Jossen]

Sunset A_n-toric variety







Sunset graphs toric variety $X_{p^2}(A_n)$ [Verrill]

The sunset graph polynomial

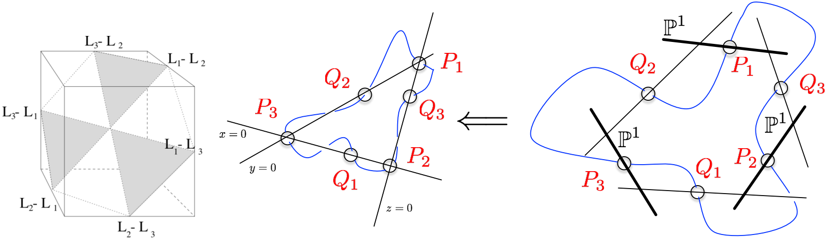
$$\mathcal{F}_n^{\ominus} = x_1 \cdots x_n \left(\left(\sum_{i=1}^{n} m_i^2 x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) - p^2 \right)$$

is a character of the adjoint representation of A_{n-1} with support on the polytope generated by the A_{n-1} root lattice

- The Newton polytope Δ_n for $\mathcal{F}_n^{\ominus}$ is reflexive with only the origin as interior point
- The toric variety $X(A_{n-1})$ is the graph of the Cremona transformations $X_i \to 1/X_i$ of $\mathbb{P}^{n-1}$
  $X(A_{n-1})$ is obtained by blowing up the strict transform of the points, lines, planes etc. spanned by the subset of points $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, ..., $(0, \dots, 0, 1)$ in $\mathbb{P}^{n-1}$

Two-loop Sunset toric variety $X(A_2)$

$$(m_1^2x_1+m_2^2x_2+m_3^2x_3)(x_1x_2+x_1x_3+x_2x_3)=p^2x_1x_2x_3$$



- The toric variety is $X(A_2)=Bl_3(\mathbb{P}^2)=dP_6$ blown up at 3 points
- The subfamily of anticanonical hyperspace is non generic
  The combinatorial structure of the **NEF** partition describes precisely the mass deformations
- True for all n

Sunset graphs pencils of variety $\mathcal{X}_{p^2}(A_n)$ [Verrill]

For $p^2 \in \mathbb{P}^1$ we define the pencil in the ambient toric variety $X(A_{n-1})$

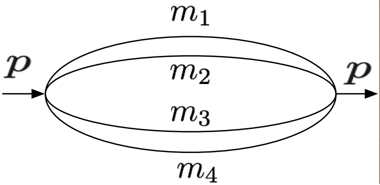
$$\mathcal{X}_{p^2}(A_{n-1}) = \{(p^2, \underline{x}) \in \mathbb{P}^1 \times X(A_{n-1}) | x_1 \cdots x_n \left(\sum_{i=1}^{n} m_i^2 x_i\right) \left(\sum_{i=1}^{n} \frac{1}{x_i}\right) - p^2 x_1 \cdots x_n = 0\}$$

The fiber at $p^2 = \infty$ is $\mathcal{I}_n = \{x_1 \cdots x_n = 0\}$

Since $\mathcal{I}_n$ is linearly equivalent to the anti-canonical divisor of $X(A_{n-1})$ the family has trivial canonical divisor: We have a family of (singular) Calabi-Yau $n-2$-fold

This is specific to this family of associated with root lattice of A_n

The Iterative fibration



The Iterative fibration

The sunset family $\left(\sum_{i=1}^{n} m_i^2 x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right) - p^2 = 0$ is birational to a complete intersection variety in $\mathbb{P}^n$

$$\frac{1}{x_0} + \sum_{i=1}^{n} \frac{1}{x_i} = 0; \qquad p^2 x_0 + \sum_{i=1}^{n} m_i^2 x_i = 0$$

Obviously $X(A_{n-1})$ is obtained from $X(A_{n-2})$ with the substitutions

$$\frac{1}{x_{n-1}} \to \frac{1}{x_{n-1}} + \frac{1}{x_n}; \qquad m_{n-1}^2 x_{n-1} \to m_{n-1}^2 x_{n-1} + m_n^2 x_n$$

$X(A_{n-1})$ is fibrered over $X(A_1) = \mathbb{P}^1$ with generic fibers $X(A_{n-2})$

$$X(A_{n-2}) \to X(A_{n-1}) \to X(A_1) = \mathbb{P}^1$$

The Iterative fibration

The geometric phenomenon at work that the n-loop sunset corresponds to a family of Calabi-Yau $(n-1)$-folds each of which is a double cover of the (rational) total space of a family of $(n-1)$-loop sunset Calabi-Yau $(n-2)$-folds.

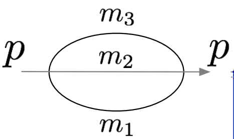
At the level of the integrals this

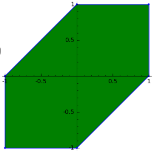
$$I_n^{\ominus}(p^2, \underline{m}^2) = \int_0^{+\infty} I_{n-1}^{\ominus}\left(p^2, \underline{m}^2, (m_{n-1}^2 + t^{-1} m_n^2)(1+t)\right) d\log t$$

and for the classical period

$$\pi_n^{\ominus}(p^2, \underline{m}^2) = \frac{1}{2i\pi} \int_{|t|=1} \pi_{n-1}^{\ominus}\left(p^2, \underline{m}^2, (m_{n-1}^2 + t^{-1} m_n^2)(1+t)\right) d\log t$$

This construction allows to understand the geometry and build the PF operator for all loop orders [Doran, Novoseltsev, Vanhove]





The pencil of sunset elliptic curve

$$\mathcal{X}_{p^2}(A_2) = \{(p^2, \underline{x}) \in \mathbb{P}^2 \times X(A_2) | (m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3)(x_1 x_2 + x_1 x_3 + x_2 x_3) = p^2 x_1 x_2 x_3\}$$

The fibers types are

- Generic case $m_1 \neq m_2 \neq m_3$

$$I_2(0) + I_6(\infty) + 4 I_1(\mu_i); \qquad \mu_i = (\pm m_1 \pm m_2 \pm m_3)^2$$

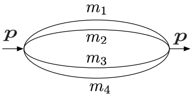
- single mass $m_1 = m_2 = m_3 \neq 0$: modular curve $X_1(6)$

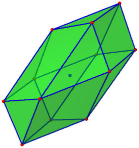
$$I_2(0) + I_6(\infty) + I_3(m^2) + I_1(9m^2)$$

The Feynman integral is an elliptic dilogarithm [Bloch, Kerr, Vanhove]

$$H^2(\mathbb{P}^2 \backslash \{x_1 x_2 x_3 = 0\}, X_\ominus, \mathbb{Q}(2))$$

The 3-loop case : pencil of $K3$





$$\mathcal{X}_{p^2}(A_3) := \{(p^2, \underline{x}) \in \mathbb{P}^1 \times X(A_3) | \left(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3 + m_4^2 x_4\right)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_4}\right) = p^2\}$$

Generic anticanonical $K3$ hypersurface in the toric threefold X_{Δ° has Picard rank 11

The physical locus for the sunset has at least Picard rank 16

masses	fibers	Mordell-Weil	Picard rank
(m_4, m_1, m_2, m_3)	$8I_1 + 2I_2 + 2I_6$	2	16
$(m_4 = m_1, m_2, m_3)$	$8I_1 + I_4 + 2I_6$	2	17
$(m_4, m_1, m_2 = m_3)$	$4I_1 + 4I_2 + 2I_6$	1	17
$(m_4 = m_1, m_2 = m_3)$	$4I_1 + 2I_2 + I_4 + 2I_6$	1	18
$(m_4 = m_1 = m_2, m_3)$	$8I_1 + I_4 + 2I_6$	3	18
$(m_4, m_1 = m_2 = m_3)$	$4I_1 + 4I_2 + 2I_6$	2	18
$(m_4 = m_1 = m_2 = m_3)$	$4I_1 + 2I_2 + I_4 + 2I_6$	2	19

$|Pic| = 19$ motive of an elliptic 3-log $H^3(\mathbb{P}^3 \backslash \Pi_4, X_4, \mathbb{Q}(3))$ [Bloch, Kerr, Vanhove]

The Picard-Fuchs operator

$$
\begin{tikzcd}
& L_3 & \\
L_4 \arrow[ur, "m_4=m_1"] & & L_4 \arrow[ul, "m_4=m_1"'] \\
& L_5 \arrow[ul, "m_3=m_1"] \arrow[ur, "m_3=m_4"'] & \\
& L_6 \arrow[u, "m_2=m_1"] &
\end{tikzcd}
$$

$$L_r = (\alpha \frac{d}{dp^2} + \beta) \circ L_{r-1}$$

The Picard-Fuchs operators for the Feynman integral for general parameters $m_4 \neq m_1 \neq m_2 \neq m_3$

$$L_6 = \sum_{r=0}^{6} q_r(s) \left(\frac{d}{dp^2}\right)^r$$

is order 6 and degree 25

$$q_6(p^2) = \tilde{q}_6(p^2) \times \prod_{\epsilon_i=\pm 1} (p^2 - (\epsilon_1 m_1 + \epsilon_2 m_2 + \epsilon_3 m_3 + \epsilon_4 m_4)^2)$$

with $\tilde{q}_6(p^2)$ degree 17 with apparent singularities

The 4-loop case : pencil of CY 3-fold

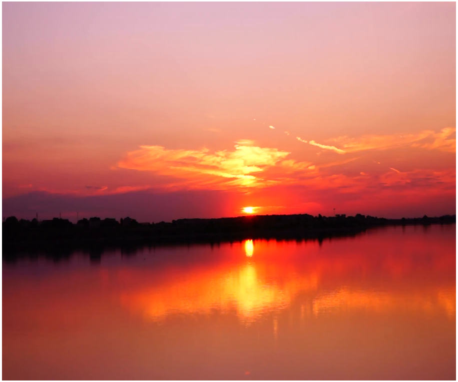
$$\mathcal{X}_{p^2}(A_4) := \{(p^2, \underline{x}) \in \mathbb{P}^1 \times X(A_4) | \left(m_1^2 x_1 + \cdots + m_5^2 x_5\right) \left(\frac{1}{x_1} + \cdots + \frac{1}{x_5}\right) = p^2\}$$

This gives a pencil of nodal Calabi-Yau 3-fold

For a (small or big) resolution $\hat{W}$ is

- $h^{12}(\hat{W}) = 5$ for the 5 masses case : 30 nodes
- $h^{12}(\hat{W}) = 1$ for the 1 mass case $m_1 = \cdots = m_5$: 35 nodes
- $h^{12}(\hat{W}) = 0$ for $p^2 = m_1 = \cdots = m_5 = 1$: rigid case birational to the Barth-Nieto quintic
    - $I_5^{\ominus}(1, \ldots, 1) = 48\zeta(2)L(f, 2)$ [Broadhurst]
    - f weight 4 and level 6 modular form $f = (\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$
    - This L-series is precisely the one for $H^3(X(A_4), \mathbb{Q}_\ell)$ [Verrill]
    - Functional equation $L(f, s) \propto L(f, 4 - s)$
    - Again we have a manifestation of Deligne's conjecture

Sunset Mirror Symmetry



Sunset local Gromov-Witten invariants

The sunset Feynman integral takes the expression [Bloch, Kerr, Vanhove]

$$I_3^{\ominus}(p^2) = \pi_3^{\ominus}(p^2)\left(3R_0^2 + \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell>0\\(\ell_1,\ell_2,\ell_3)\in\mathbb{N}^3\backslash(0,0,0)}} \ell(1-\ell R_0) N^{\rm loc.}_{\ell_1,\ell_2,\ell_3}\prod_{i=1}^{3} Q_i^{2\ell_i}\right).$$

- The Kähler parameters are $Q_i = m_i^2 e^{R_0}$
- With $\pi_3^{\ominus}(p^2) = \frac{d}{dp^2} R_0$

$$R_0 := \int_{|x_1|=|x_2|=|x_3|=1} \log(p^2 - \phi_3^{\ominus}) \prod_{i=1}^{3} \frac{d\log x_i}{2\pi i}$$

- The classical period is

$$\pi_3^{\ominus}(p^2, \underline{m}^2) = \int_{|x_1|=|x_2|=|x_3|=1} \frac{1}{p^2 - \phi_3^{\ominus}} \prod_{i=1}^{3} \frac{d\log x_i}{2\pi i}$$

Sunset local Gromov-Witten invariants

Considering the Yukawa coupling for the sunset elliptic curve

$$\mathcal{Y}_\ominus = \int_{E_\ominus} \Omega_\ominus \wedge \nabla_{p^2 \frac{d}{dp^2}} \Omega_\ominus; \qquad \Omega_\ominus := \mathrm{Res}_{X_\ominus} \left(\frac{1}{\mathcal{F}_\ominus(\underline{x})} \frac{dx_1 dx_2}{x_1 x_2} \right)$$

This descends from the local Yukawa coupling Hori-Vafa 3-fold obtained as the total space of the anticanonical line bundle on del Pezzo dP_6 for the sunset

$$Y_3^\ominus := \{p^2 - (\xi_1^2 x_1 + \xi_2^2 x_2 + \xi_3^2 x_3)(x_1^{-1} + x_2^{-1} + x_3^{-1}) + uv = 0, (\underline{x}, u, v) \in \mathbb{P}^2 \times (\mathbb{C}^*)^2\}$$

Leading to this generating series for the *local* Gromov-Witten invariants $N_{\underline{\ell}}^{\mathrm{loc.}}$

$$\sum_{i,j} d_i d_j \int_Y \Omega \wedge \nabla_{0,i,j}^3 \Omega \sim \frac{\mathcal{Y}_\ominus(p^2)}{\pi_2^\ominus(p^2)^3}$$

$$= 6 - \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell>0 \\ (\ell_1,\ell_2,\ell_3)\in\mathbb{N}^3\backslash(0,0,0)}} \ell^3 N_{\ell_1,\ell_2,\ell_3}^{\mathrm{loc.}} \prod_{i=1}^3 Q_i^{\ell_i}$$

Sunset local Gromov-Witten invariants

$N_{\underline{\ell}}$ are given by the BPS counting integer number of rational curves

$$N^{\text{loc.}}_{\ell_1,\ell_2,\ell_3} = \sum_{d|\ell_1,\ell_2,\ell_3} \frac{1}{d^3} n_{\frac{\ell_1}{d},\frac{\ell_2}{d},\frac{\ell_3}{d}} \,.$$

- Agree with the BPS invariants computed using the refined holomorphic anomaly [Huang, Klemm, Poretschkin]
- Unfortunately for higher n these *local* Gromov-Witten invariants are difficult to compute

The mirror map between CY treat all the Kähler on the same footing but the p^2 is special as a physical variables as the base parameter for the pencils construction

Sunset relative Gromov-Witten invariants

Luckily they are totally equivalent to *relative* degree d Gromov-Witten invariants $N_\beta^{\mathrm{rel.}}(dP_6, D)$ where D a smooth NEF anti-canonical divisor

[Van Garrel, Graber, Ruddat]

$$N_\beta^{\mathrm{rel.}}(dP_6, D) = (-1)^{\beta\cdot D}(\beta\cdot D)N_\beta^{\mathrm{loc.}} \qquad \beta\in H_2(dP_6,\mathbb{Z})$$

By virtue of the particular nature of our family of sunset integral this relation holds to all loop orders and the mirror symmetry can be recasted as a manifestation to the Fano / Landau-Ginzburg mirror symmetry

Sunset Landau-Ginzburg mirror

The LG superpotential is the sunset graph polynomial

$$w = \mathcal{F}_n^{\ominus}(\underline{x}) = x_1 \cdots x_n \left(p^2 - \phi_n^{\ominus}(\underline{x})\right)$$

is homogeneous of degree n in $\mathbb{P}^{n-1}$ therefore the central charge is $c = 3(n-2)$ in agreement with the statement that $\mathcal{X}_{p^2}(A_{n-1})$ is a Calabi-Yau $n-2$-fold

The advantages is that the relative invariants have a purely algebraic expression

We can use the localisation technique of [Tseng, You] for genus 0 invariant case. Details to appear in [Doran, Novoseltsev ,Vanhove]

Conclusion

- We have put forward the new relation between Feynman integrals and mirror symmetry between Fano / LG model
- It is a new result that all the sunset Feynman integrals compute the genus 0 relative Gromov-Witten invariants
- The Fano/LG mirror symmetry shed some light on the Sunset Feynman integral and provide a uniform approach to all orders
- The iterative structure induces the double cover construction should lead to a proof of the Broadhurst conjectures for the L-function values

Generic Feynman graphs is more intricate

- For Feynman graph with $\deg(\mathcal{F})_\Gamma = L$ in $\mathbb{P}^n$ with $n > L + 1$ we do not have a Calabi-Yau
  - Multiple potential LG models?
  - Relations with the Doran-Harder-Thompson construction for Tyurin degenerations cf. [Doran (strings math 2015, Sanya)]