Stability and nonlinear PDEs in mirror symmetry

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Outline

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1. Mirror symmetry
2. The special Lagrangian and deformed Hermitian-Yang-Mills equations
3. Variational Approach to dHYM
4. Connections with algebraic geometry, and stability conditions

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- Namely, in certain limits, CY manifolds should admit fibrations by special Lagrangian tori, and the mirror CY is constructed by dualizing the fibers.

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- The SYZ proposal gives a geometric mechanism for this equivalence using T-duality and a real Fourier-Mukai transform.

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- The equation is the *deformed Hermitian-Yang-Mills* equation.

Definition (dHYM equation)

A holomorphic line bundle $L \to (V, \omega|_V)$ *solves the deformed Hermitian-Yang-Mills equation if it admits a smooth hermitian metric* h *such that* $F(h) = -\partial\overline{\partial}\log h$ *solves*

$$\mathrm{Im}(e^{-i\hat{\theta}}(\omega - F)^{\dim_{\mathbb{C}} V}) = 0, \qquad \hat{\theta} \in \mathbb{R}$$

$$\mathrm{Re}(e^{-i\hat{\theta}}(\omega - F)^{\dim_{\mathbb{C}} V}) > 0$$

PDEs and Algebraic Geometry

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Conjecture (Folklore)

A Lagrangian submanifold $L \hookrightarrow (X, \omega)$ *(resp. holomorphic line bundle* $L \to (X, \omega)$*) can be deformed to a special Lagrangian (resp. admits a hermitian metric solving the dHYM equation) if and only if* $[L]$ *is stable in* $D^{\pi}Fuk(X)$ *(resp.* $D^{b}Coh(X)$*) in the sense of Bridgeland.*

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- NOTE: Bridgeland stability conditions are not known to exist in general, so this conjecture is really two conjectures.

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This is equivalent to the equation for line bundles by taking $\mathfrak{a} = -c_1(L)$ and $\alpha = -\sqrt{-1}F(h)$.

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- We can rewrite the dHYM equation in terms of the relative endomorphism K of $T^{1,0}(X)$ given by

$$K := \omega^{j\bar{k}}\alpha_{\bar{k}\ell}\frac{\partial}{\partial z^j}\otimes dz^\ell.$$

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- The dHYM equation is

$$\Theta_\omega(\alpha) = \hat{\theta}$$

for a constant $\hat{\theta} \in (-n\frac{\pi}{2}, n\frac{\pi}{2})$, determined ($\mod 2\pi$) by cohomology:

$$e^{\sqrt{-1}\zeta} = e^{\sqrt{-1}\hat{\theta}}.$$

The deformed Hermitian-Yang-Mills equation

- Recall that a graph $\mathbb{R}^n \ni x \mapsto (x, \nabla f(x)) \in \mathbb{R}^{2n} = \mathbb{C}^n$ is special Lagrangian if and only if

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- The dHYM equation is therefore a natural, complex (and global version) of the graphical sLag equation.

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$$\mathcal{H}_{\hat{\theta}} = \{\varphi \in C^{\infty}(X, \mathbb{R}) : |\Theta_{\omega}(\alpha_{\varphi}) - \hat{\theta}| < \frac{\pi}{2}\}$$

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- From now on assume $\hat{\theta} > (n-1)\frac{\pi}{2}$.

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Theorem (Collins-Yau, Solomon)

$\delta\mathcal{J}(\varphi)(\psi)$ *integrates to a well-defined function* $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ *with the property that*

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A Variational Approach

- Associated to the Riemannian structure is a notion of *geodesics*.(to be discussed. But first, the upshot.)
- Consider the following 1-form on $\mathcal{H}$. Given $\psi \in T_{\varphi}\mathcal{H}$ define

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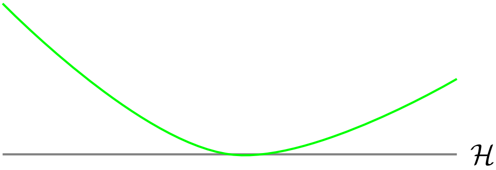

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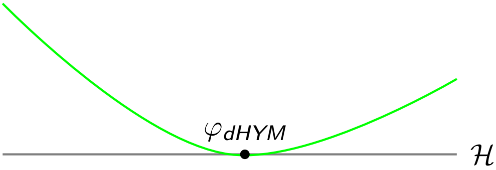

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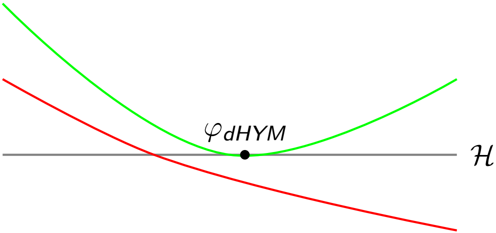

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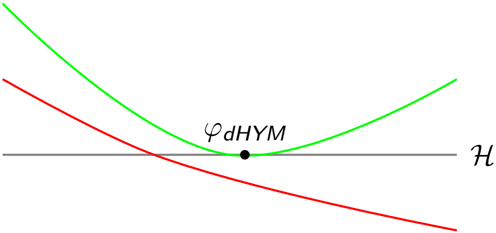

The distinguishing feature is *the slope* of $\mathcal{J}$ near "$\partial\mathcal{H}$".

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The geodesic equation in $\mathcal{H}$: Step 1

To introduce the geodesic equation it is convenient to introduce the manifold

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Lemma

Suppose $\varphi_0, \varphi_1 \in \mathcal{H}$. A geodesic segment $\varphi(x,s) \in \mathcal{H}$ with $\varphi(x,0) = \varphi_0$, $\varphi(x,1) = \varphi_1$ is equivalent (by setting $s = -\log|t|$) to a function $\varphi : \mathcal{X} \to \mathbb{R}$ which is S^1 invariant (ie. $\varphi(x,t) = \varphi(x,|t|)$) and solving

$$\mathrm{Im}\left[e^{-\sqrt{-1}\hat{\theta}}\left(\pi_X^*\omega + \sqrt{-1}\left(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}\right] = 0 \qquad (1.1)$$

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Some other interesting functionals

In addition to $\mathcal{J}$, there is a whole S^1 worth of interesting functionals, generated by the functional $CY_{\mathbb{C}} : \mathcal{H} \to \mathbb{C}$ whose derivative is

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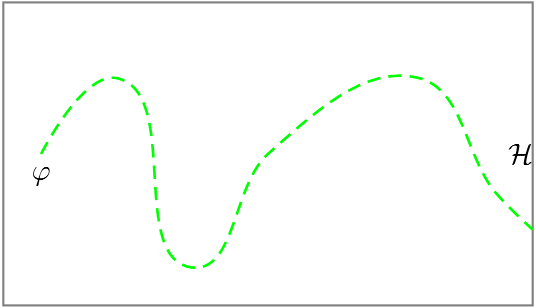

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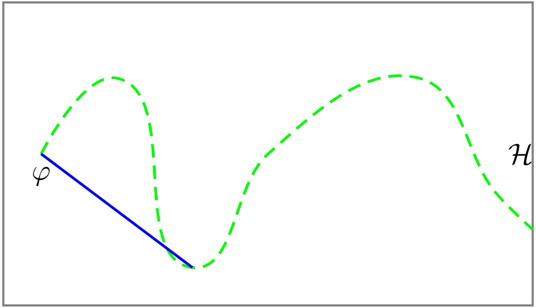

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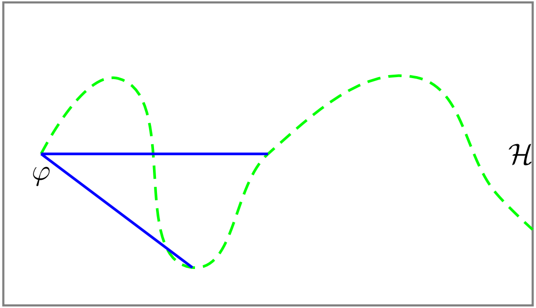

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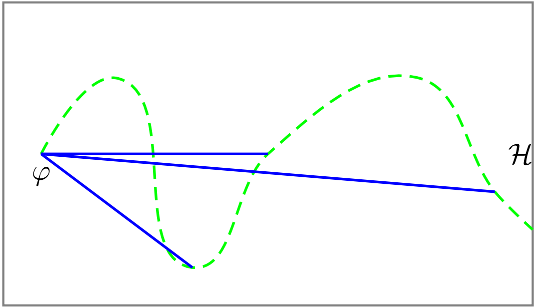

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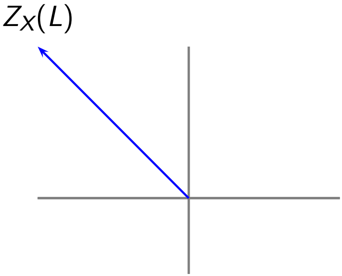

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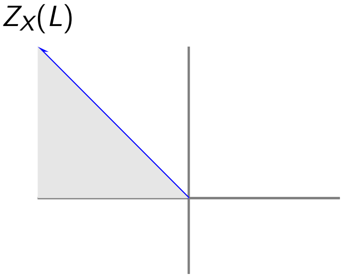

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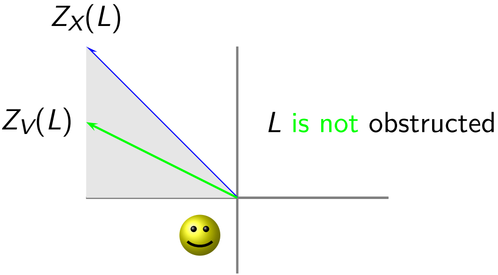

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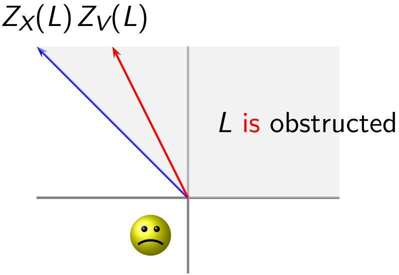

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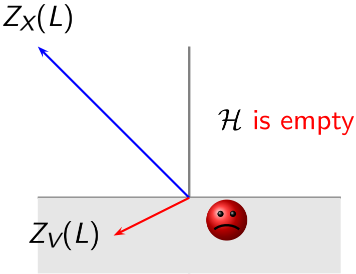

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Sufficiency?

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- In the large radius limit, and even for higher rank bundles, this reduces to the Donaldson-Uhlenbeck-Yau Theorem.

Relation to the Bridgeland stability framework

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- That is, an assignment to $E \in D^b Coh(X)$ of a phase $\varphi(E)$, so that

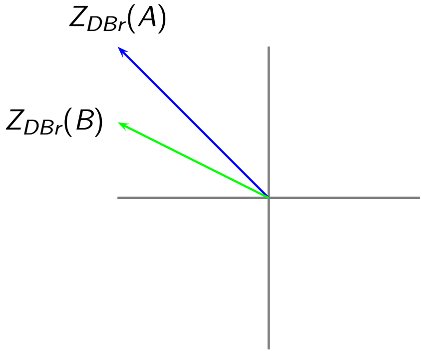
$$Z_{DBr}(E) \in \mathbb{R}_{>0} e^{\sqrt{-1}\varphi(E)}$$

Relation to the Bridgeland stability framework

Definition

An object $A \in D^bCoh(X)$*, satisfying* $\varphi(A) \in (0, \pi]$ *is* Bridgeland stable *if for any object* B *with* $\varphi(B) \in (0, \pi]$ *with* $A \twoheadrightarrow B$ *we have*

$$\varphi(B) > \varphi(A).$$



Stable

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An object $A \in D^b Coh(X)$, satisfying $\varphi(A) \in (0, \pi]$ is Bridgeland stable *if the following holds. For any object B with $\varphi(B) \in (0, \pi]$, with $A \twoheadrightarrow B$ we have*

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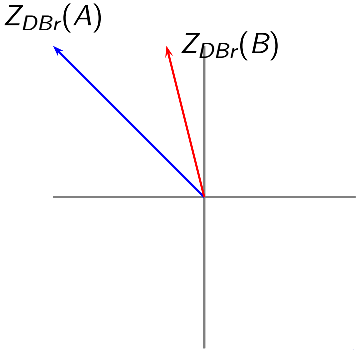

Unstable

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The obstructions we obtained by our variational framework can be regarded as arising from the surjections

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$$Z_{DBr}(L \otimes \mathcal{O}_V) \neq Z_V(L)$$

in general.

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- In this case

$$\hat{\theta}(L) = \text{ Winding Angle } \gamma(t).$$

provided $\gamma(t)$ does not pass through the origin.

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Theorem (Collins-Xie-Yau)

Assume $\dim X = 3$. *If* $L \to X$ *has a solution of dHYM with* $\hat{\theta}(L) \in (\frac{\pi}{2}, \frac{3\pi}{2})$, *then* $\gamma(t)$ *does not pass through the origin. This follows from the Chern number inequality*

$$\int_X \omega^3 \int_X ch_3(L) < 3\left(\int_X ch_2(L).\omega\right)\left(\int_X ch_1(L).\omega^2\right).$$

We expect to have an inequality in dimension n involving $ch_1, \dots, ch_n$.

Applications in symplectic geometry

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- $W : Y \to \mathbb{C}$ is a holomorphic function called the super-potential.
- A holomorphic line bundle $L \to (X, \omega)$ with a metric $h \in \mathcal{H}$ is transformed to a *almost calibrated* (a.c) Lagrangian $\hat{L} \subset Y$.

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- The problem of finding special Lagrangian sections of Y is equivalent to solving the graphical sLag equation with certain singular boundary data.
- Under this correspondence the $\mathcal{J}$ functional becomes a functional on the space of almost calibrated (a.c) Lagrangians originally discovered by Solomon.
- Our degenerations/geodesics in the space of metrics in $\mathcal{H}$ give rise to degenerations/geodesics in the space of a.c. Lagrangians.

Applications in symplectic geometry

- For example, consider $\mathcal{O}(-k) \to \mathbb{P}^1$, and consider the degeneration corresponding to

$$\mathcal{I} = (Z_1^2 Z_2) + (t) \subset \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}[t].$$

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- In this case $Y = (0,2) \times \mathbb{R}/\mathbb{Z}$.

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- Understand fully the obstructions we produce in Landau-Ginzburg models, and relate them to the Fukaya category of (Y, W).
- Prove the conjecture!

Thank You!