Dirac quantization and Maxwell anomaly in string theory

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Based on

- [1805.02772] with Y. Tachikawa
- [1905.08943][To appear] with C.T. Hsieh, Y. Tachikawa

Introduction

String theory contains several higher form gauge fields, such as RR p-form fields C :

$$C = C_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \quad \text{(locally)}$$

Their fluxes, $F = dC$, are often said to have integral periods due to Dirac quantization condition:

$$\int_{\text{cycle}} F \in \mathbb{Z} \quad \text{(integer)}$$

Introduction

Is the Dirac quantization condition really satisfied in string theory?

Introduction

Orientifold $\mathrm{O}p$-plane has RR charges given by $\pm 2^{p-5}$

This means that the integral of $F = dC$ around the O-plane is given by

$$\int_{\text{around } Op} F = \pm 2^{p-5}$$

This is not integer for $p < 5$

It seems that Dirac quantization condition is violated.

Introduction

Remark 1
This is not a problem of the existence of singularity.

Example: $(\mathbb{R}^6/\mathbb{Z}_2) \times \mathbb{R}^4 \to AdS^5 \times (S^5/\mathbb{Z}_2)$

- The manifold $(S^5/\mathbb{Z}_2) = \mathbf{RP}^5$ is completely smooth.
- But the flux $\int_{\mathbf{RP}^5} F = \frac{1}{4}$ mod 1, is not integer.

Even in some smooth geometry, fluxes are fractional and in particular nonzero.

Introduction

Remark 2
RR fluxes are classified by K-theory or its variants.
(Generalized cohomology)

[Moore-Witten,1999]
[Bergman-Gimon-Sugimoto,2001]
[Freed,2001]

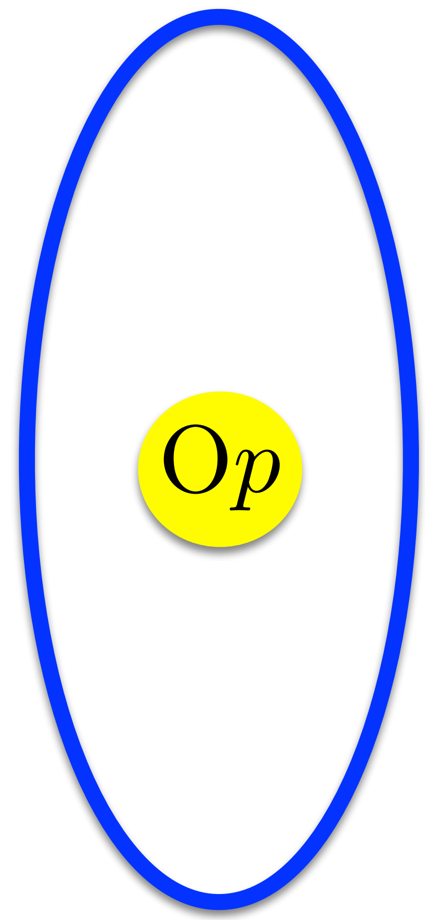
However, the argument of Dirac quantization which I'm going to review is independent of such classification framework.

Let us recall the argument of Dirac quantization and how anomalies change it.

Contents

Dirac quantization

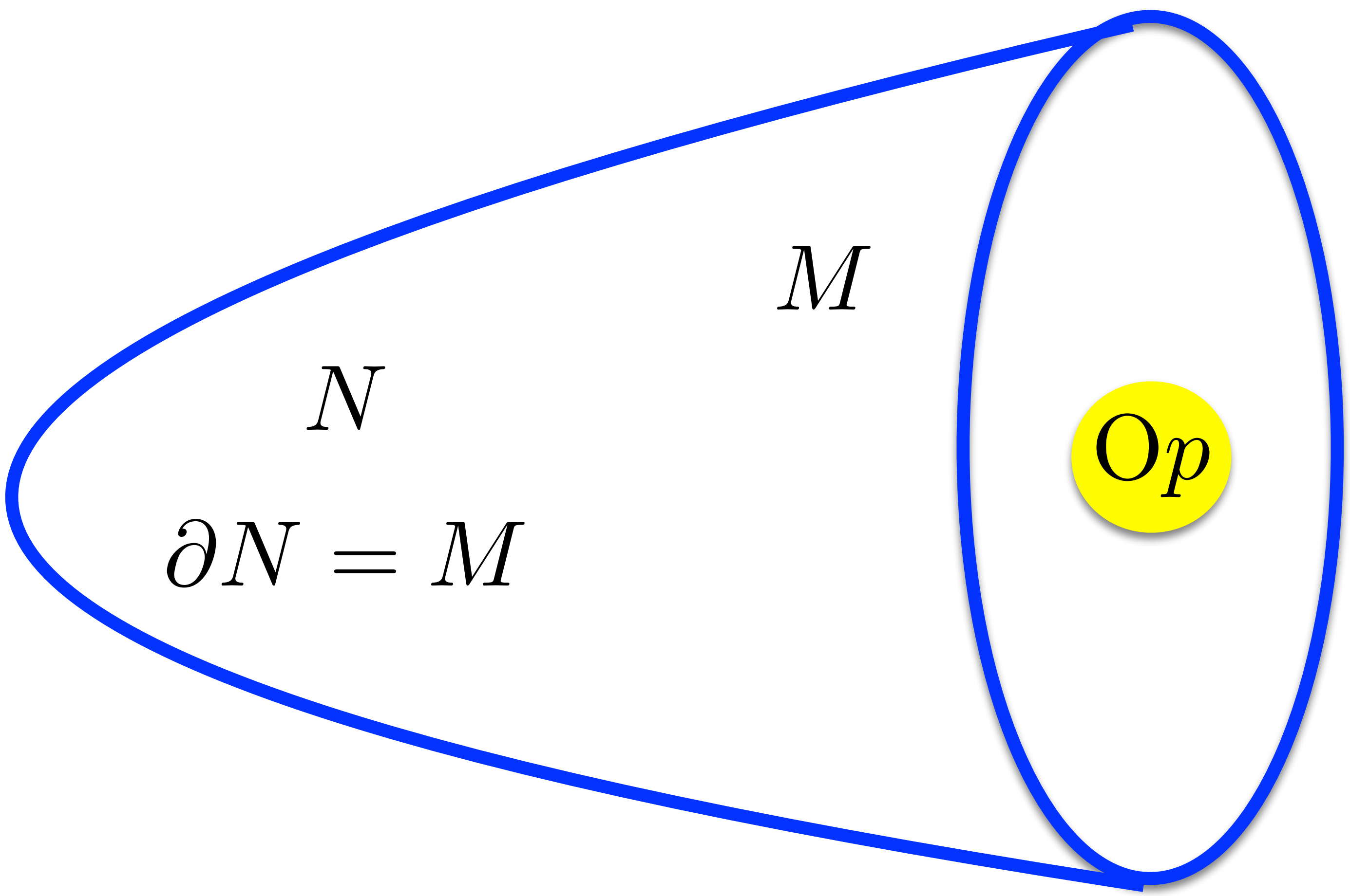
D-brane
worldvolume M



Coupling to RR field $\exp(2\pi i \int_M C)$

C : RR-field

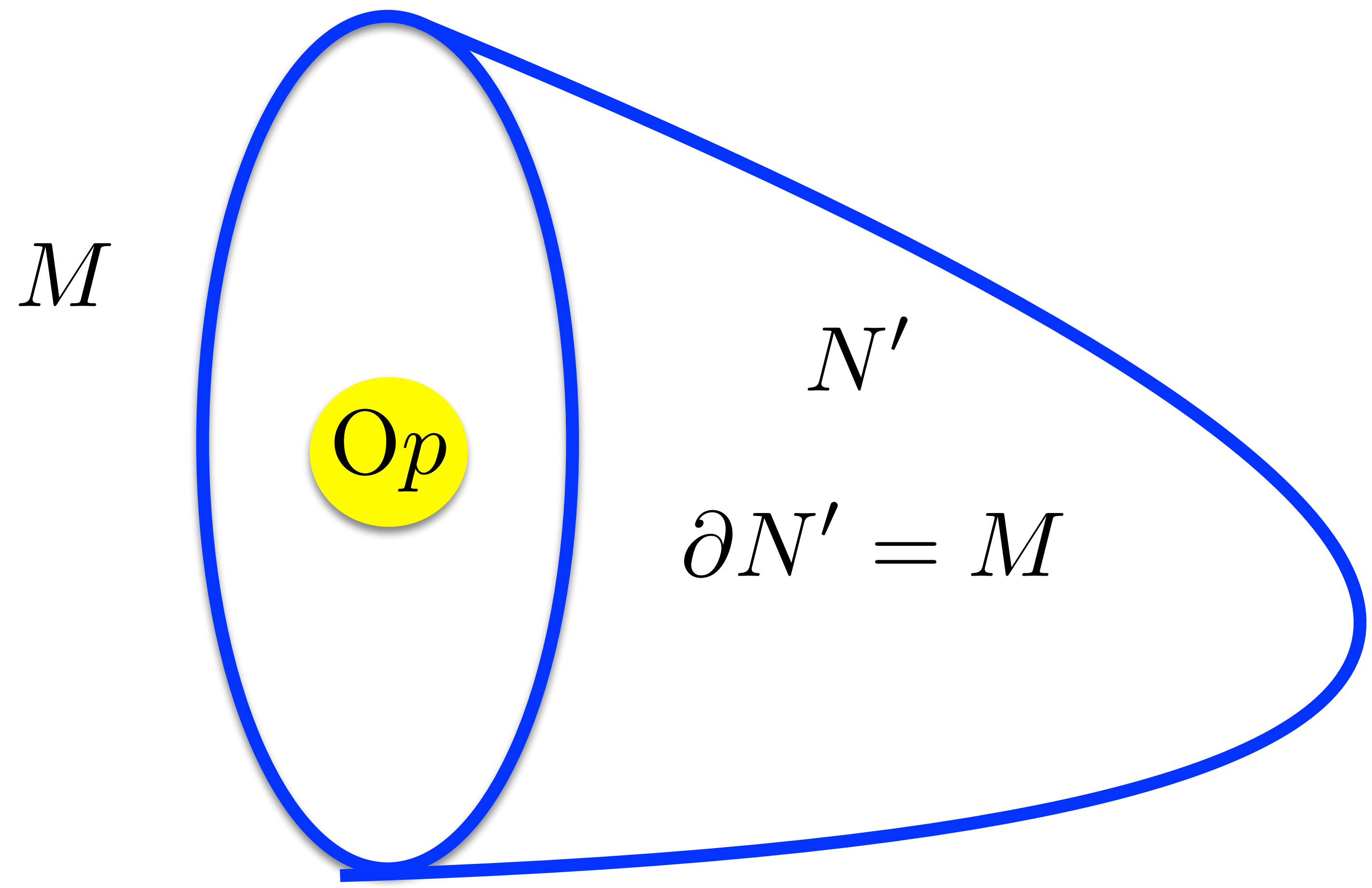
Dirac quantization



Coupling to RR field $\exp(2\pi i \int_M C) = \exp(2\pi i \int_N F)$

C : RR-field $\quad F = dC$

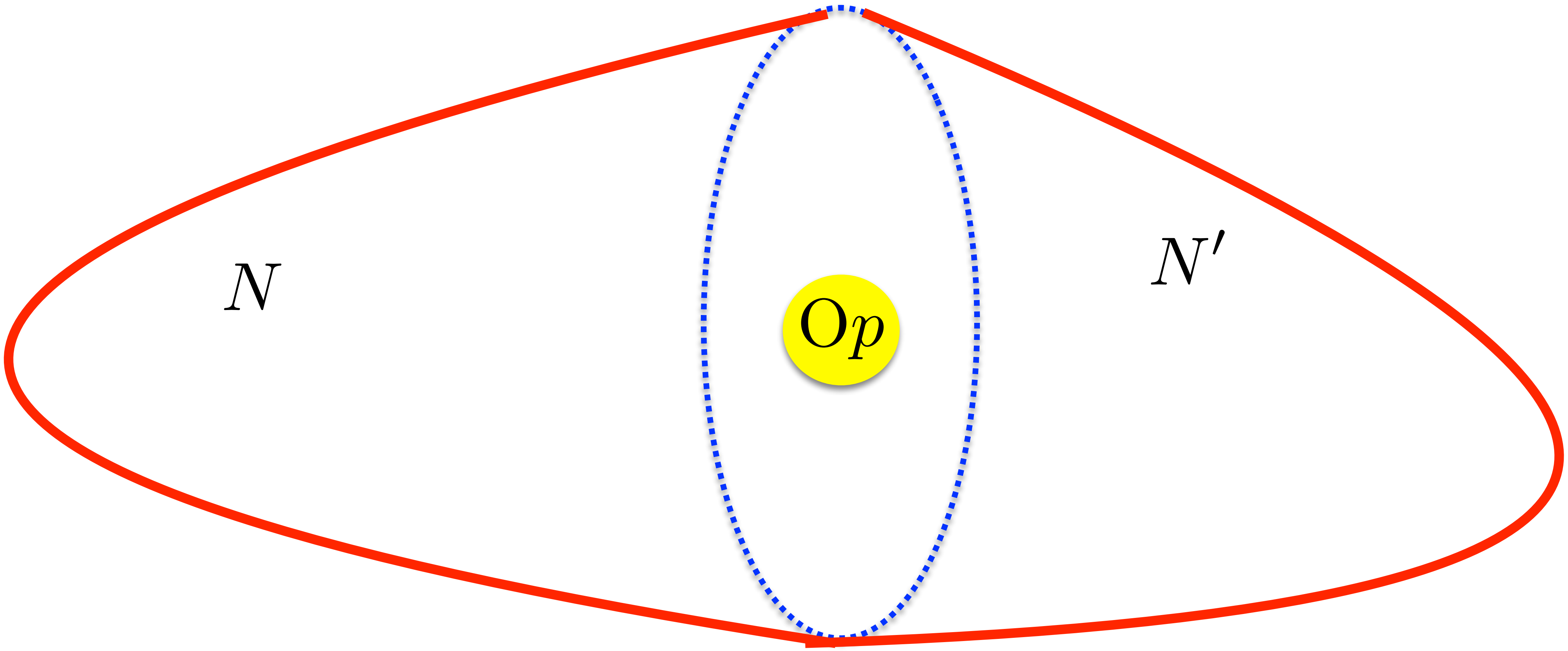
Dirac quantization



Coupling to RR field $\exp(2\pi i \int_M C) = \exp(2\pi i \int_{N'} F)$

C : RR-field $\quad F = dC$

Dirac quantization



$$\frac{\exp(2\pi i \int_N F)}{\exp(2\pi i \int_{N'} F)} = \exp(2\pi i \int_X F)$$

$X = N \cup \overline{N}'$: closed manifold

Dirac quantization

$$\int_X F \notin \mathbb{Z}$$ (Violation of Dirac condition on closed manifolds X)

$$\longrightarrow \quad \exp(2\pi i \int_X F) \neq 1$$

$$\longrightarrow \quad \exp(2\pi i \int_N F) \neq \exp(2\pi i \int_{N'} F)$$

The coupling depends not only on the world-volume M but also on how to extend to N such that $\partial N = M$.

$\exp(2\pi i \int_M C)$ is ambiguous

Partition function

How can we fix this ambiguity of the RR coupling?

The total partition function of D-brane worldvolume is

$$Z[M] \exp(2\pi i \int_M C)$$

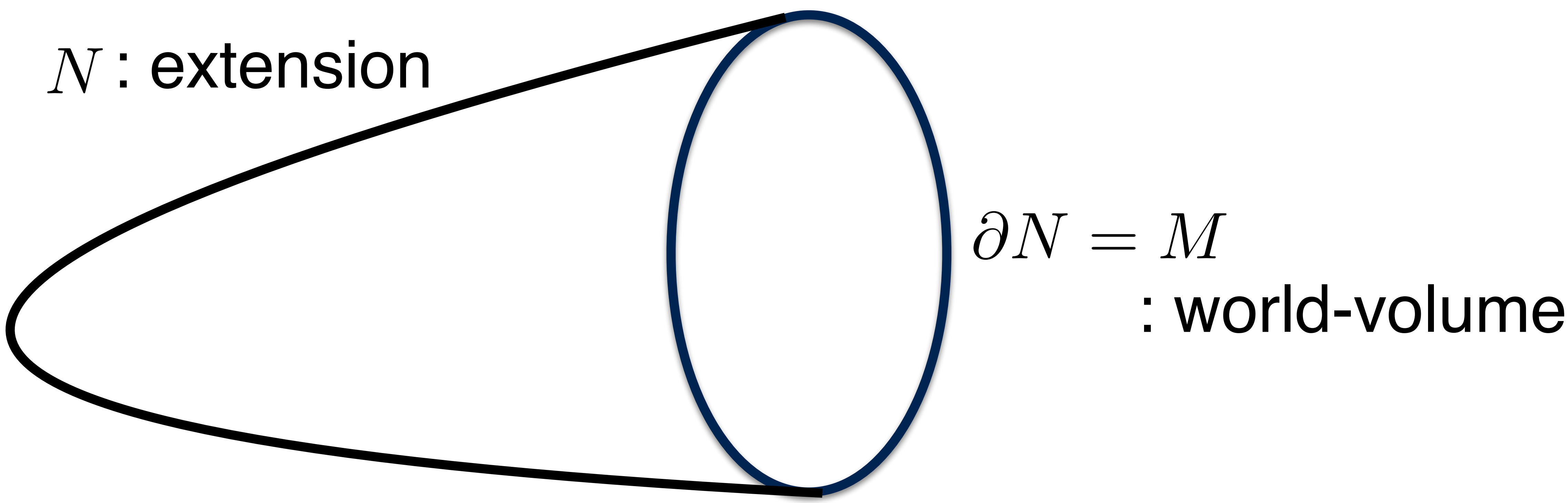
The partition function of worldvolume fields (scalars, fermions, gauge fields)

Coupling to RR-field

Partition function

The partition function $Z[M]$ is also ambiguous by anomaly.

It depends on how we extend M to N



Characterization of anomaly

$Z[N]$: partition function depending on N

$$\frac{Z[N]}{Z[N']} = Z[X] \in \mathrm{U}(1)$$: pure phase

($X = N \cup \overline{N'}$)

$Z[X] \neq 1$: anomaly (Explicit example later.)

Modern understanding of anomaly:

Dependence of the partition function on how to extend M to N.

This understanding was developed both from string, math, and cond.matt. by many people.

[Witten,Freed,Wen,...]

Anomaly free condition

The consistency of the world-volume of D-branes is controlled not by $Z[X]$ or $\int_X F$, but by the product

$$Z[X]\exp(2\pi i\int_X F)$$

Anomaly free condition: (Generalized Dirac quantization)

$$Z[X]\exp(2\pi i\int_X F) = 1$$

X : closed manifold

Shifted flux quantization

Fluxes are not quantized to be integers, but

$$\int F \in A + \mathbb{Z}$$

$$A = -\frac{1}{2\pi i} \log Z[X] \in \mathbb{R}/\mathbb{Z}$$

A : quantity which is controlled by the anomaly of the world-volume theory of D-brane.

O-plane

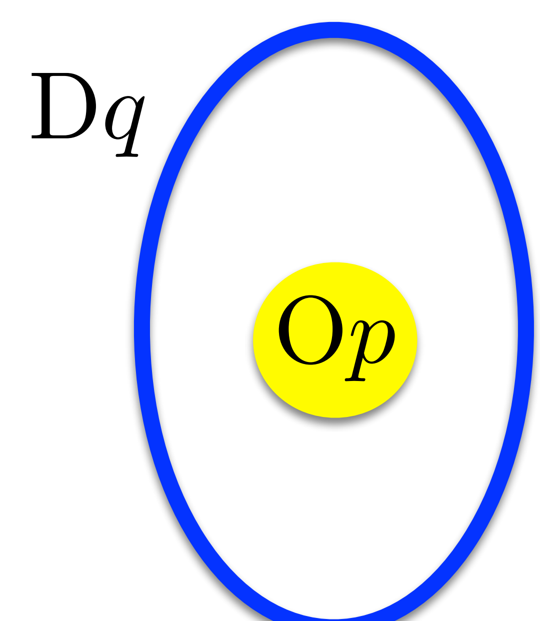
Let us return to O-plane background geometry

$$\int_{\mathbf{RP}^{8-p}} F = \pm 2^{p-5} \qquad p < 5$$

There must be an anomaly in the worldvolume of Dq-brane with $q = 6 - p$

$$\mathrm{O}p \longleftrightarrow \mathrm{D}q$$

Dirac pair $q = 6 - p$



O-plane

World-volume fermions (gauginos) have an anomaly

General formula for fermion anomaly

$$Z[X] = \exp(-2\pi i \eta(X))$$ [Witten,2015]

η : Atiyah-Patodi-Singer eta invariant

Careful computation shows that in the O-plane geometry

$$|\eta(X = \mathbf{RP}^{8-p})| = 2^{p-5}$$

The sign is such that it cancels the anomaly for O^{+}

[Witten,1996,2016]
[Tachikawa-KY,2018]

O-plane

Let us recall two types of O-planes: $\mathrm{O}^{\pm}$-planes

O^{+} : charge $+2^{p-5}$, cancelled by the fermion anomaly!

O^{-} : charge -2^{p-5} , **???**

What explains the difference?

$$(+2^{p-5}) - (-2^{p-5}) = 2^{p-4}$$

This is not integer if $p < 4$

D-brane matter content

D-brane matter content	Anomaly
Scalar	$\int F$
Fermion	η
Maxwell	???

Maxwell theory must have anomaly for the consistency of string theory.

Contents

O3-D3 system

$\mathrm{O}p^{\pm}$ charge difference

$$(+2^{p-5}) - (-2^{p-5}) = 2^{p-4}$$

becomes fractional if $p < 4$

The first nontrivial case: $p = 3$

Let us consider D3-brane in O3-plane background $\mathbf{RP}^5$.

Maxwell on D-brane

The action of the Maxwell at the differential form level :

$$S = \frac{1}{2e^2}\int (dA + B') \wedge \star (dA + B') + 2\pi i \int C' \wedge (dA + B')$$

A : dynamical U(1) Maxwell field

B : background NS 2-form field

C : background RR 2-form field

(B', C') and (B, C) are slightly different.

Maxwell with background

$$S = \frac{1}{2e^2} \int (dA + B') \wedge \star (dA + B') + 2\pi i \int C' \wedge (dA + B')$$

In modern Language, the Maxwell theory has electric and magnetic 1-form symmetries.

[Gaiotto-Kapustin-Seiberg-Willett, 2016]

In that terminology,

B' : background for electric 1-form symmetry

C' : background for magnetic 1-form symmetry

Maxwell with background

At the level of differential form,

$$S \supset 2\pi i \int_M C' \wedge (dA + B')$$

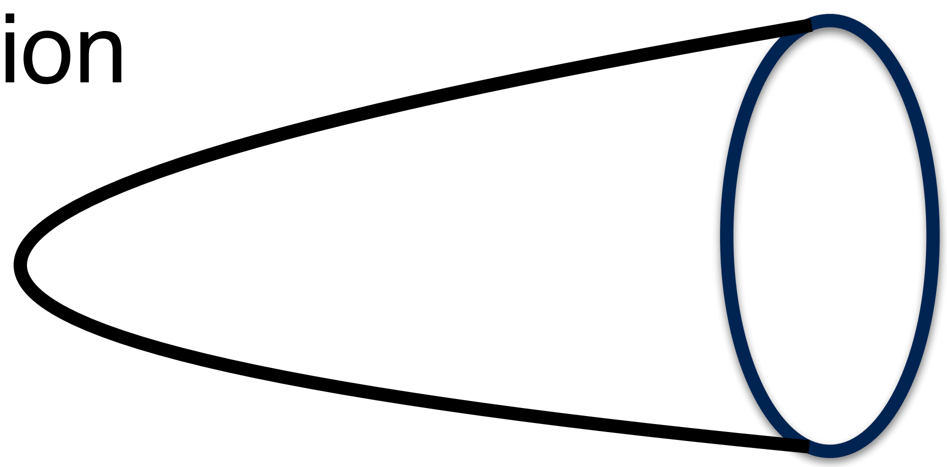
It is not gauge invariant under $C' \to C' + d\alpha$

The violation of gauge invariance implies anomaly.
Let us reformulate it in the modern understanding.

Maxwell with background

We take an extension N such that $\partial N = M$

N : extension



$\partial N = M$
: worldvolume

$$\int_M C' \wedge (dA + B') \to \int_N dC' \wedge (dA + B')$$: gauge invariant

(At the level of differential form)

The fact that we need such extension N : anomaly

Anomaly

On a closed manifold X , at the level of differential form,

$$\int_X dC' \wedge (dA + B') = \int_X dC' \wedge B'$$

The anomaly of electric-magnetic 1-form symmetries:

$$\int_X dC' \wedge B'$$

[Gaiotto-Kapustin-Seiberg-Willett, 2016]

Anomaly

Let's compute the anomaly in the O3-background

$S^5/\mathbb{Z}_2 = \mathbf{RP}^5$: real projective space

$$dB', dC' \in H^3(\mathbf{RP}^5, \widetilde{\mathbb{Z}}) = \mathbb{Z}_2$$

More careful treatment beyond differential form level based on Cheeger-Simons theory gives that

$$\int_X dC' \wedge B' = \frac{1}{2} nm \mod 1$$

(torsion pairing)

$$(dB', dC') = (m, n) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

Anomaly

$$dB', dC' \in H^3(\mathbf{RP}^5, \widetilde{\mathbb{Z}}) = \mathbb{Z}_2$$

$$\int_X dC' \wedge B' = \begin{cases} \frac{1}{2} & (dB', dC') = (1,1) \\ 0 & (dB', dC') = (0,0), (0,1), (1,0) \end{cases}$$

One background (1,1) gives the anomaly.
The other three don't have anomaly.

Notice: $2^{p-4} = \frac{1}{2}$ $\quad \mathrm{O}3^{\pm}$ charge difference

Four O3-planes

There are four types of O3-planes.

$\mathrm{O3}^-$: charge $-\dfrac{1}{4}$

$\mathrm{O3}^+, \widetilde{\mathrm{O3}}^-, \widetilde{\mathrm{O3}}^+$: charge $+\dfrac{1}{4}$

[Witten, 1998]

Comparing to the previous anomaly, we identify

(dB', dC')	$(1,1)$	$(0,1), (1,0), (0,0)$
O-plane	$\mathrm{O3}^-$	$\mathrm{O3}^+, \widetilde{\mathrm{O3}}^-, \widetilde{\mathrm{O3}}^+$

Remark on SL(2,Z) invariance

SL(2,Z) of Type IIB string acts

$\mathrm{O3}^-$: invariant

$\mathrm{O3}^+, \widetilde{\mathrm{O3}}^-, \widetilde{\mathrm{O3}}^+$: permuted

$(dB', dC') = (1,1)$ for $\mathrm{O3}^-$. So we must shift

$$dB' = dB + w^3 \qquad dC' = dC + w^3$$

$w^3 \in H^3(\mathbf{RP}^5, \widetilde{\mathbb{Z}})$: generator

- (B, C) are the usual NS RR 2-forms acted by SL(2,Z).
- The reason for the shift of B' : Maxwell field is spin^c

Contents

O2-D4 system

For O2-plane, the charge difference between $\mathrm{O}2^{\pm}$ is

$$(+2^{p-5}) - (-2^{p-5}) = 2^{p-4} = \frac{1}{4}$$

However, $H^n(\mathbf{RP}^6)$ for $0 < n < 6$ are just $\mathbb{Z}_2$ or zero.

The product $\int_X dC' \wedge B'$ can give only $\frac{1}{2}$ mod 1

Ordinary cohomology product is not enough.

Remark: difference between Op^- and $\widetilde{Op}^-$ is $\frac{1}{2}$

M-theory

I don't know how to treat Maxwell anomaly directly in Type IIA string, but fortunately there is a lift to M-theory.

O2-D4 system $\longrightarrow$ OM2-M5 system (Two copies of OM2)

An OM2 is an orbifold $\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{R}^3$

We need to compute the anomaly of M5 in the presence of OM2 background.

M-theory

Self-dual 2-form field B on M5-brane

↓

Maxwell field A on D4-brane

The Maxwell anomaly is described by the anomaly of self-dual 2-form field.

Self-dual 2-form anomaly

Schematically, the action of self-dual 2-form field is

$$S \sim \frac{2\pi}{2} \int \frac{1}{2}(dB + C) \wedge \star(dB + C) + iC \wedge (dB + C)$$

B : self-dual 2-form dynamical field
C : 3-form background (M-theory 3-form potential)

The anomaly is schematically

$$\int \frac{1}{2} C dC + \text{gravitational}$$

This can be obtained at the perturbative level.

Self-dual 2-form anomaly

For OM2, the relevant manifold for anomaly is $\mathbf{RP}^7$

$$dC \in H^4(\mathbf{RP}^7, \mathbb{Z}) = \mathbb{Z}_2$$

$$\int_{\mathbf{RP}^7} CdC = \begin{cases} \frac{1}{2} & dC = 1 \\ 0 & dC = 0 \end{cases}$$

So, very roughly, by “dividing by 2”, we get

$$\int_{\mathbf{RP}^7} \frac{1}{2} CdC = \begin{cases} \frac{1}{4} & dC = 1 \\ 0 & dC = 0 \end{cases}$$

This explains the $\mathrm{O}2^{\pm}$ difference $2^{p-4} = \frac{1}{4}$

Quadratic refinement

However, “dividing by 2” is not an allowed operation in integral cohomology theory.

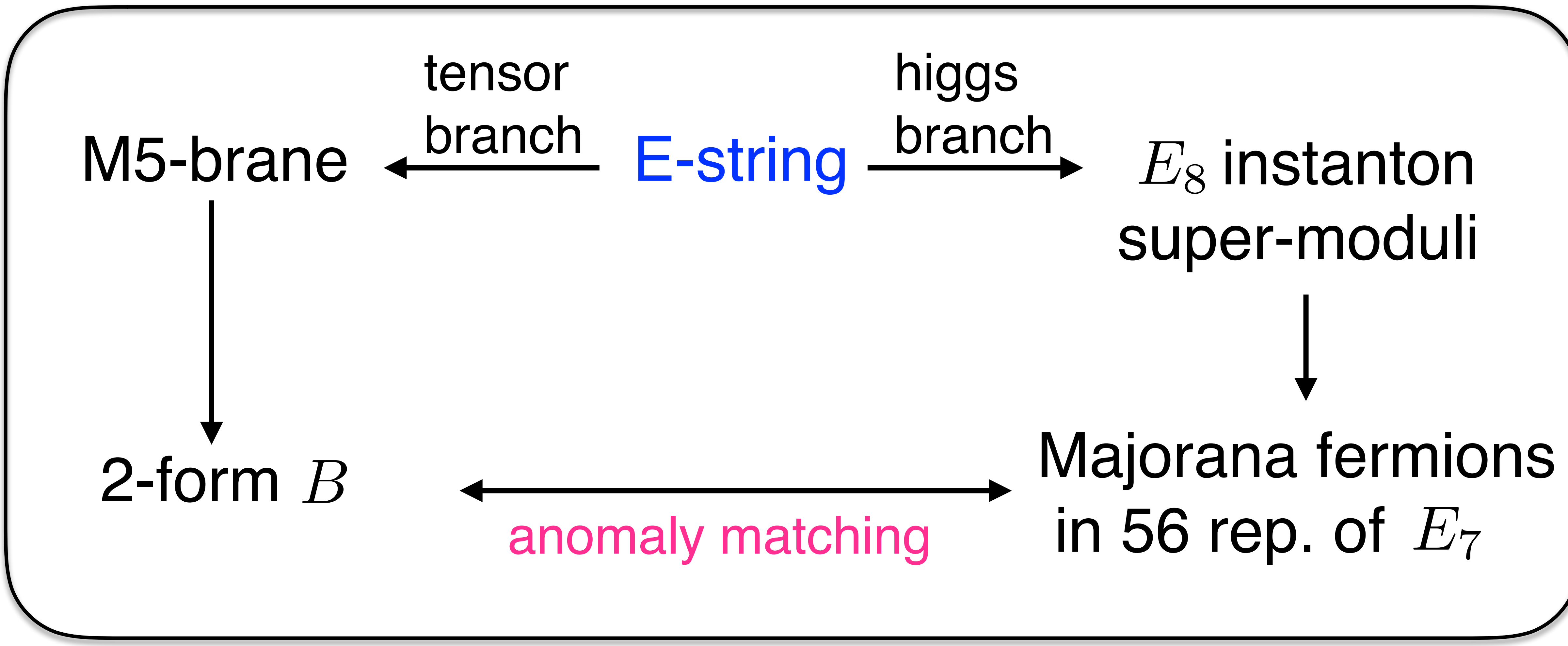
$$\int CdC \xrightarrow{\text{“dividing by 2”}} \int \frac{1}{2} CdC$$

$$\frac{1}{2} \mod 1 \longrightarrow \frac{1}{4} \text{ or } \frac{3}{4} \ ? \mod 1$$

We need a precise quadratic refinement of (differential) cohomology pairing.

Anomaly matching

We use the following fact:



(Remark: Instanton breaks $E_8 \to E_7$)

Quadratic refinement

By using the matching, the quadratic refinement can be defined precisely by the eta invariant of Majorana fermions in 56 dim. rep. of E_7

$$\int \frac{1}{2} C dC + \text{gravitational} \;=\; \frac{1}{2}\eta(56 \text{ of } E_7)$$

This formula reproduces the known fractional charges of OM2 and its generalizations.

[Sethi,1998]
[Bergman-Hirano, 2009]
[Aharony-Hashimoto-Hirano-Ouyang,2009]

E7?

We originally had 3-form background C, but somehow it becomes E_7 background. What happened?

In M-theory it is known that 3-form field is (topologically) Chern-Simons 3-form of E_8

$$C \sim \mathrm{CS}(E_8)$$

[Horava-Witten,1995]
[Witten,1996]
[Diaconescu-Freed-Moore,2013]

A topological fact underlying this is

$$BE_8 \sim K(\mathbb{Z}, 4)$$

up to very high homotopy group dimensions.

E7?

E_7 still has the same property

$$BE_7 \sim K(\mathbb{Z}, 4)$$

up to very high dimensions enough for our purposes.

$$C \sim \mathrm{CS}(E_7)$$

Under this identification, we can use the formula

$$\int \frac{1}{2} C dC + \text{gravitational} \;=\; \frac{1}{2}\eta(56 \text{ of } E_7)$$

Unique quadratic refinement

Remark
Even without using E_7 formulation, the quadratic refinement is uniquely defined by perturbative anomaly polynomial due to the fact

$$\Omega_7^{\mathrm{spin}}(K(\mathbb{Z},4)) = 0$$

[Stong,1985]
[Monnier-Moore,2018]

(More generally, cobordism groups classify global anomalies.)

[Kapustin et al,2014]
[Freed-Hopkins,2016]
[KY,2018]

Anomaly of EM duality

The self-dual 2-form field can be also used to formulate the electric-magnetic duality of 4d Maxwell theory by putting the theory on T^2 fiber bundle

6d 4d

$$T^2 \to F \to M$$

SL(2,Z) duality of 4d Maxwell has an anomaly.

[Seiberg-Tachikawa-KY,2018]
[Hsieh-Tachikawa-KY,2019]

This has applications to generalization of O3-planes, called S-folds constructed in F-theory.

[Garcia-Etxebarria-Regalado,2015]
[Aharony-Tachikawa,2016]

$$(\mathbb{R}^6 \times T^2)/\mathbb{Z}_k \times \mathbb{R}^4$$

Contents

Summary

- Fractional parts of fluxes are determined by the anomaly of world-volume theory.

$$\int F \in A + \mathbb{Z} \qquad\qquad A \in \mathbb{R}/\mathbb{Z} \ \text{: anomaly}$$

- Maxwell anomaly plays an important role in the consistency of D-branes in O-plane (and other) backgrounds.

Future directions

- Direct understanding of D4-O2 anomaly in Type IIA without uplifting to M-theory.

- D5-O1 and D6-O0 Maxwell anomalies do not follow from 6d self-dual 2-form field, so they are much more subtle.

$$2^{p-4} = \begin{cases} \frac{1}{8} & p=1 \quad \text{: O1 plane} \\ \frac{1}{16} & p=0 \quad \text{: O0 plane} \end{cases}$$